

Thermal equilibrium states: from the Quantum to the Classical world

joint work with N. Drago and C.J.F. Van de Ven
Based on ArXiv.2406.12342

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5 July 2024



Question : How can we introduce equilibrium states for infinite volume systems?

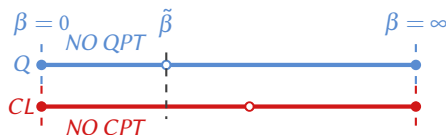
Question : Are equilibrium states preserved after decoherence?

Question : What can we say about finite temperature phase transitions?

Answer ! KMS states (Generalize Gibbs) (★)

Answer ! $Q_j^\Gamma \leftarrow$ Quantization maps
Classical Obs. $\rightarrow$ Quantum Obs. (★★)

Answer ! If $\beta < \tilde{\beta}$ No CPT $\Rightarrow$ No QPT
 N.Drago, C.J.F. Van de Ven, L.P. (2024)



(★) R. Kubo, P.C. Martin, J. Schwinger ('57-59); G. Gallavotti, E. Verboven ('67)

(★★) F.A. Berezin, CMP (75); S. Murro, C.J.F. Van de ven, MPAG (2022)

Quantization

What was possible some months ago?

💡 It all started with:

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No quantization map satisfying Dirac's and naive requirements does exist...

Solution Approximate quantizations $\rightarrow \boxed{Q_j : C(\mathbb{S}^2) \rightarrow M_{2j+1}(\mathbb{C})} \quad (j \in \mathbb{Z}_+ / 2 \text{ Particle's spin})$

Dirac-Rieffel-Groenwold:

$$\|(2j+1)[Q_j(a), Q_j(b)] - iQ_j(\{a, b\})\| \xrightarrow{j \rightarrow \infty} 0$$



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Note: $C(\mathbb{S}^2)$ functions of a single particle's angle; $M_{2j+1}(\mathbb{C})$ particle's spin matrices



More quantization

What can be done now!



Setting Approximation of cold atoms $\rightarrow$ Lattice $\mathbb{Z}^d$, for every point x a particle

Models Many physically relevant models e.g. **Quantum Ising**, **Heisenberg**, etc...

$\rightarrow$ **Our result:**

$$Q_j^{\mathbb{Z}^d} : C(\times_{x \in \mathbb{Z}^d} \mathbb{S}^2) \longrightarrow \overline{\bigcup_{\Lambda \in \mathbb{Z}^d} \bigotimes_{x \in \Lambda} M_{2j+1}(\mathbb{C})}$$

i.e. :

We derived a full working quantization for infinite volume systems

The quantization procedure survives the Thermodynamic limit

Equilibrium states

What was possible some month ago



Λ : Limited regions $\xrightarrow{\text{Gibbs states}}$ Hamiltonians h_Λ , $H_\Lambda = Q_j^\Lambda(h_\Lambda)$

$$\langle a_\Lambda \rangle_{\text{CL}} = \frac{1}{Z_\Lambda^{\text{CL}}(\beta)} \int_{\times_\Lambda \mathbb{S}^2} \frac{d\Omega_\Lambda}{(2\pi)^{|\Lambda|}} e^{-\beta h_\Lambda} a_\Lambda(\Omega_\Lambda), \quad \langle A_\Lambda \rangle_Q = \frac{1}{Z_\Lambda^Q(\beta)} \text{Tr} \left[e^{-\beta Q_j^\Lambda(h_\Lambda)} A_\Lambda \right]$$

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$\mathbb{Z}^d$: Whole lattice $\rightarrow \nexists h_{\mathbb{Z}^d}$ or $Z_{\mathbb{Z}^d}^{\text{CL}}(\beta) = Z_{\mathbb{Z}^d}^Q(\beta) = +\infty$

Solution KMS states:

$$\omega_{\text{CL}}(\{a, b\}) = i\beta \omega_{\text{CL}}(\delta(a)b), \quad \omega_Q(A\tau_{i\beta}(B)) = \omega_Q(BA)$$

$\rightarrow$ Equivalent to other equilibrium conditions **Gibbs** ($\star$), **DLR** ($\star\star$):

($\star$) Bratteli O. Robinson D.W. ('97) ($\star\star$) N. Drago, C.J.F. Van de Ven, LMP (2022)

$\rightarrow$ Extensions $\beta < 0$ ($\circ$) and/or $\Lambda \uparrow \mathbb{Z}^d$ ($\circ\circ$):

($\circ$) Abragam and Proctor PR ('57-58) ($\circ\circ$) O. Bratteli, Robinson D.W. ('97)

Preservation of equilibrium

Steps:



- 1 We start with our infinite volume quantum system $\overline{\bigcup_{\Lambda \in \mathbb{Z}^d} \bigotimes_{x \in \Lambda} M_{2j+1}(\mathbb{C})}$
- 2 We heat up the system up to temperature $1/\beta$ (but keep it quantum!)
- 3 We input energy to the system: $j \uparrow \infty, Q_j^{\mathbb{Z}^d}(f) \sim f$

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Question : What happens to the equilibrium state $\omega_j^{\text{Q-KMS}}$?

Answer : *Theorem (N. Drago, C.J.F. Van de Ven, L.P.)*

$$\boxed{\omega_j^{\text{Q-KMS}} \xrightarrow{*, j \rightarrow \infty} \omega^{\text{CL-KMS}}}$$

→ Equilibrium states converge to Equilibrium states

Phase transition characterization

New exciting result about Phase Transitions



- (★) *Theorem* (N. Drago, C.J.F. Van de Ven, L.P.)
 - New (and First) condition for Absence of CPT with classical KMS
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(★★) We only need $\beta < \tilde{\beta}$ and $\|\varphi\|_s < \infty \rightarrow$ same conditions of the classical system !

It follows : $\beta < \tilde{\beta}, \|\varphi\|_s < \infty \implies$

Quantization and De-quantization do not change the thermal phase (★★★)

(★★★) With our hypothesis

Conclusions

Summary and Outlooks



🏆₁ Extension of $Q_j \rightarrow Q^{\mathbb{Z}_d}_j : \mathfrak{A}^{CL}_{\mathbb{Z}_d} \rightarrow \mathfrak{A}^Q_{\mathbb{Z}_d}$ i.e. It works for the whole lattice

🏆₂ j Semi-classic parameter $\omega_j^{KMS-Q} \circ Q_j \xrightarrow{*, j \rightarrow \infty} \omega^{KMS-CL}$

🏆₃ $\beta \in [0, \tilde{\beta}[\varphi])$ e $\|\varphi\|_s < \infty$ with φ Classical potential $\implies$

1. Absence of Classical Phase Transitions
2. Absence of Quantum Phase transitions

Conclusion $\boxed{\text{Absence of CPT} \implies \text{Absence of QPT}}$

Outlooks :

🏆₁ Absence of QPT $\implies$ Absence of CPT ?

🏆₂ Any results for Existence of Phase Transitions ?

🏆₃ More realistic models e.g. Bose-Einstein