

On classical aspects of Bose-Einstein condensation

based on arXiv:2411.02626

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Introduction



Q: Can a **Bose-Einstein condensate** survive some classical limit ?

$$\omega_{\text{QUANTUM}}^{\rho} \circ Q_h^B \xrightarrow{\text{weak-}^*} \omega_{\text{CLASSICAL}}^{\alpha}$$

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- This formalism is useful for treating thermodynamical equilibrium for infinite volumes

Result: Definition of classical KMS states for commutative Weyl algebras

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A: To answer the first question: yes (for free systems)

Weyl algebras and notation

Weyl algebras construction



- Non interacting particles $\longrightarrow E := \mathbb{L}^2(\mathbb{R}^v) \xrightarrow{\text{second quantized}} \mathbb{F}(E) := \bigoplus_{n \in \mathbb{N}} E \otimes_s \cdots \otimes_s E$
- $(E, h\sigma)$ real, symplectic space

$$h \in [0, +\infty), \quad \sigma(f, g) := \text{Im}\{\langle f, g \rangle\}$$

semiclassical parameter



Symplectic form



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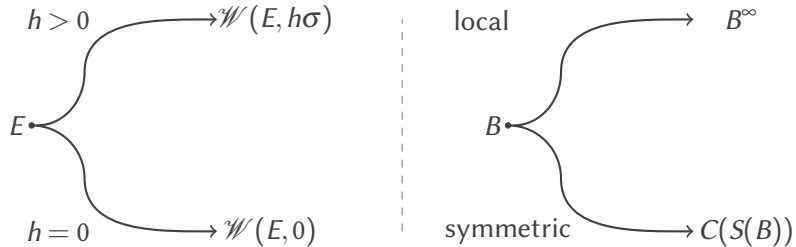
- $\Delta(E, h\sigma) := \mathbb{C}\text{-span}\{W^h(f), f \in E\}$, $*$ -algebra with relations

$$W^h(f)W^h(g) = e^{-\frac{ih}{2}\sigma(f,g)}W^h(f+g), \quad (W^h(f))^* = W^h(-f), \quad W^h(f)W^h(-f) = I$$

Interpretation of the two worlds

Weyl algebras construction

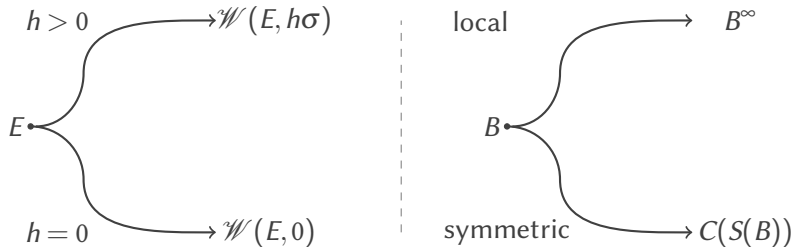
- E is the single particle quantum Hilbert space



Interpretation of the two worlds

Weyl algebras construction

- E is the single particle quantum Hilbert space



→ Classical dynamic is induced by the Quantum one - on the single particle

$$\tau_t^h(W^h(f)) = W^h(e^{iHt}f) \xrightarrow{h \rightarrow 0} W^0(e^{iHt}f) := \tau_t^0(W^0(f))$$

$h \neq \hbar$ h is related to the macroscopic density, not to the single particle scale constants in H

Weak KMS states

Weak-classical KMS condition



Def. (Weak KMS states) Let $\beta > 0$, $\delta: \Delta(G, 0) \rightarrow C(E)$ weak derivation $\omega \in C^2$ satisfies the weak (δ, β) –KMS condition if

$$\beta \omega(\{c, d\}) = \omega(d\delta(c)), \quad c, d \in \Delta(G, 0)$$

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Hyp. $\delta(W^0(f)) = \Phi_0(iHf)W^0(f)$ with $H > 0$, self-adjoint and H^{-s} trace class ($s > 0$)

Th. (P., arXiv:2411.02626 (2024)) there exists a unique (δ, β) -weak KMS

$$\omega(W^0(f)) = \exp\left\{-\frac{1}{2\beta} \langle f, H^{-1}f \rangle\right\}$$

Prop. ω satisfies the classical (δ_ω, β) KMS condition on $\pi_\omega(\mathscr{W}(E, 0))''$, where $\delta_\omega: D(\delta_\omega) \rightarrow \pi_\omega(\mathscr{W}(E, 0))''$ is a true (unbounded) derivation

Definition of the quantization map

Abstract quantization map

Def. *Strict deformation quantization*:

- $(\Delta(E, 0), \{ \cdot, \cdot \}), \{ Q_h : \Delta(E, 0) \rightarrow \Delta(E, h\sigma) \}_{h \in [0, +\infty)}$ injective net of maps satisfying

$$- Q_h(1) = I, \quad Q_h(a)^* = Q_h(a^*)$$

$$- \lim_{h \rightarrow 0} \| Q_h(a) \|_h = \| a \|_0, \quad \lim_{h \rightarrow 0} \| Q_h(ab) - Q_h(a)Q_h(b) \|_h = 0$$

$$- \lim_{h \rightarrow 0} \left\| Q_h(\{a, b\}) - \frac{i}{h} [Q_h(a), Q_h(b)] \right\|_h = 0$$

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⚙ Our operative definition:

$$Q_h : \Delta(E, 0) \ni W^0(f) \longrightarrow e^{-\frac{h}{4} \|f\|^2} W^h(f) \in \Delta(E, h\sigma)$$

Berezin map defines a positive SDQ

Abstract quantization map

! $Q_h(W^0(f)) = w(f, h) \cdot Q_h^{\text{Weyl}}(W^0(f))$ with

$$w(f, 0) = 1, \quad w(f, h) = w(-f, h), \quad E \ni f \rightarrow w(f, h) \text{ continuous}$$

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proof: $c \in \Delta(E_0, 0) \rightarrow c = \sum_{i=1}^n z_i W^0(\sum_k \lambda_k^i f_k + \mu_k^i g_k),$

$$Q_h(c) = \sum_{i=1}^n z_i e^{-\frac{h}{4} \sum_k ((\lambda_k^i)^2 + (\mu_k^i)^2)} W^h(\lambda_k^i f_k + \mu_k^i g_k)$$

! After some lengthy computations : $\langle \phi, \pi_S(Q_h|_{E_0})(c) \psi \rangle = \langle \phi, Q_h^B(c) \psi \rangle,$

$$Q_h^B \geq 0 \xrightarrow{\pi_S\text{-faithful}} Q_h \geq 0$$

Limit of the condensate

Quantum to Classical connection



$h > 0$ Thermodynamic limit ($v \geq 3$) $\mu_L \downarrow 0$ wit fixed density $\rho > \rho_c(\beta h) \rightarrow \omega_h^{\mu, L} \xrightarrow{L \rightarrow \infty} \omega_h^\rho$

$$\omega_h^\rho(W^h(f)) = \omega_h^{0, \infty}(W^h(f)) \cdot \exp\left\{-\frac{h}{4} 2^{v+1} (\rho - \rho_c(\beta h)) \left|\int d^v x f(x)\right|^2\right\} \leftarrow \text{condensate}$$

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$$h(\rho(h) - \rho_c(h\beta)) \xrightarrow{h \rightarrow 0} \frac{\alpha}{\beta} \geq 0, \quad \text{renormalized density}$$

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Th. $\forall \alpha \geq 0$, ω_0^α satisfies the weak (δ_0, β) -KMS condition

$$D(\delta_0) = \Delta(\mathcal{S}(\mathbb{R}^v), 0), \quad \delta_0(W^0(f)) = i\Phi_0(iHf)W^0(f), \quad H = \frac{p^2}{2}$$