

Subcriticality at High Temperatures in Lattice Systems

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Recent Advances in Many-Particle Systems  - Pisa

Based on: *Comm. Math. Phys.* **406**, 163 (2025)

and ArXiv:2511.12651

9 February 2026



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- New:
1. **Convergence** of Quantum equilibrium states to Classical ones
 2. **Uniqueness of KMS states** at high temperature $\frac{1}{\beta}$ independently from the single-site dimension

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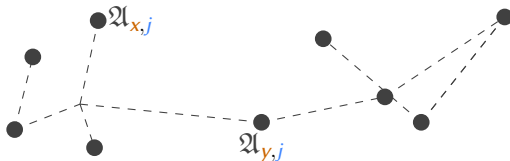
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 - We take an **infinite volume limit** $\Lambda \uparrow \Gamma$ to describe macroscopic observables
States describing equilibrium in infinite volume **can be not-unique**

The framework - quasi local algebras I

Quasi-local algebras structure

We study a system of particles sitting on fixed points of a **graph** Γ

- For every $x \in \Gamma$ there is C^* -**algebra** $\mathfrak{A}_{x,j}$ (Banach space + involution + product + C^* -norm $\rightarrow$ for our scope $M_{2j+1}(\mathbb{C})$ or $C(\mathbb{S}^2)$)

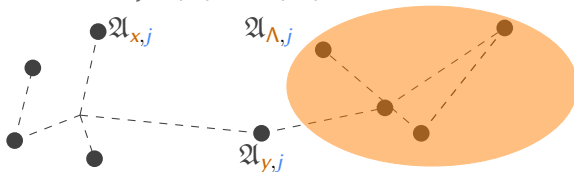


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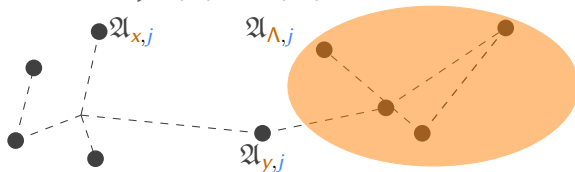
$$\mathfrak{A}_{\Lambda,j} := \bigotimes_{x \in \Lambda} \mathfrak{A}_{x,j}$$

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- For **finite volume** $\Lambda \in \Gamma$

$$\mathfrak{A}_{\Lambda,j} := \bigotimes_{x \in \Lambda} \mathfrak{A}_{x,j}$$

- Then, we construct **local** and **quasi-local** algebras

$$\mathfrak{A}_{\text{loc},j} := \bigcup_{\Lambda \in \Gamma} \mathfrak{A}_{\Lambda,j}, \quad \mathfrak{A}_j := \overline{\bigcup_{\Lambda \in \Gamma} \mathfrak{A}_{\Lambda,j}}$$

Infinite volume equilibrium - KMS states

Equilibrium states

Rmk: a state is a positive, normalized linear functional $\omega: \mathfrak{A} \rightarrow \mathbb{C}$



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- $t \rightarrow \tau_t^j: \mathfrak{A}_j \rightarrow \mathfrak{A}_j$ strongly continuous (**describes the dynamics**)
- ω_{β}^j satisfies the (τ^j, β) -**KMS boundary condition** if for all $A, B \in \mathfrak{A}_{\tau} \subset \mathfrak{A}_j$

$$\omega_{\beta}^j(A\tau_{i\beta}^j(B)) = \omega_{\beta}^j(BA)$$

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! **Finite volume** Gibbs state $\omega_{\beta, \Lambda}^j$ satisfies the Quantum/Classical KMS condition

Theorem (N.Drago, P., C.J.F. van de Ven) suppose τ_t^j/δ are "well behaved"

Take $\{\omega_\beta^j\}_{j<\infty}$ a sequence of (τ^j, β) **quantum KMS state**

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Theorem (N.Drago, P., C.J.F. van de Ven) suppose τ_t^j/δ are "well behaved"

Then there exists a **subcritical inverse temperature** β_u **uniform in** j such that for all $\beta < \beta_u$, there exists a **unique** ω_β^j **KMS state** (β_u can be computed explicitly !)

Discussion of the dynamics

Hypothesis on potential



Q: $\tau_t(A_X) = \lim_{\Lambda \rightarrow \Gamma} e^{iH_\Lambda t}(A_X)e^{-iH_\Lambda t}$ where

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X), \quad \sup_{x \in \Gamma} \sum_{X \ni x} e^{\varepsilon |X|} \|\Phi(X)\| < +\infty \text{ for some } \varepsilon > 0$$

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$$\delta(a_\Lambda) = \sum_{X \cap \Lambda \neq \emptyset} \{\varphi_X, a_\Lambda\}, \quad \sup_{x \in \Gamma} \sum_{X \ni x} e^{\varepsilon|X|} \|\varphi(X)\|_{C^1} < +\infty$$

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! The proof of $\omega_\beta^j \circ Q^j \rightarrow \omega_\beta^\infty$ **requires** $\Phi_j(X) = Q^j(\varphi(X))$

A Proof of uniqueness at high temperatures: the idea

Uniqueness of KMS states



- We can injectively map $\omega_\beta \rightarrow \underline{\omega_\beta} \in \underline{X}$, where $\underline{X}$ is a certain Banach space

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$$(1 - \underline{L_\beta})\underline{\omega_\beta} = \underline{\delta} \text{ in-homogeneous linear equation}$$

with $\underline{\delta} \in \underline{X}$ and $\underline{L_\beta} : \underline{X} \rightarrow \underline{X}$ some linear operator

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! The **novelty** of our proof lies in the **choice of $\underline{X}$**

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With $\mathcal{Y}_{j|\ell_{\Lambda}, m_{\Lambda}}$ quantized/classical spherical harmonics

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- **Decomposition in traceless components:**

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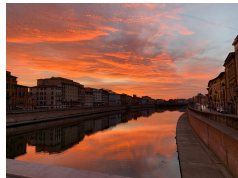
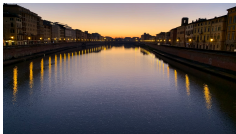
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! Future Outlooks: find an extension for Bose-Hubbard/Hubbard models

Thank you for your attention !



(Non-Exhaustive) List of references

Uniqueness of KMS states



- First use of the method: **Mayer, Kirkwood and Salzburg** - for classical correlations (Around 1945)
- **Bratteli O., Robinson D. (1997)** - notes to Section 6.2.5 contain an exhaustive list of references
- **Fröhlich J., Ueltschi D. J. Math. Phys. 56, 053302 (2015)** - revised the proof a little
- **Drago N., P. van de Ven C.J.F. Comm. Math. Phys. 406, 163 (2025)** proof independent from single-site dimension
- **Drago N., P. van de Ven C.J.F. ArXiv:2511.12651** - better estimates, broader class of potentials, new decomposition of observables

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