Algebraic Approaches to Bose-Einstein Condensation

Lorenzo Pettinari [a]

[a]Dipartimento di Matematica, Università di Trento and INFN-TIFPA and INdAM, Via Sommarive 14, I-38123 Povo, Italy

February 23, 2026

Abstract

We review two recent applications of operator-algebraic methods to the analysis of homogeneous Bose gases at finite temperature. The first part concerns the semiclassical description of Bose–Einstein condensation in the framework of Weyl C^*-algebras. In this setting, the semiclassical parameter is naturally linked to the density of the system, and the classical weak KMS condition emerges as the limit of quantum equilibrium states. We show in particular that the condensate structure is preserved in a suitable large-density regime.

The second part discusses the use of the Araki–Woods representation to formulate a systematic perturbative theory for interacting Bose gases at positive temperature. Within this representation, thermal effects are encoded directly in the field operators, allowing for a transparent implementation of Wick's theorem and the computation of damping coefficients via the Fermi Golden Rule.

The review synthesizes results obtained in two recent works presented at IQSA2025, emphasizing the conceptual role of algebraic methods in connecting semiclassical analysis, equilibrium states, and finite-temperature perturbation theory.

Keywords: Bose-Einstein condensation, C^*–algebras, W^*–algebras, Araki–Woods representations, Perturbation theory.

1 Introduction

The main object of this note is the homogeneous Bose gas [9, 33]. In both the physics and mathematical literature, a standard approach to the study of this system consists in considering an N-particle gas confined in a large but finite cubic box

$$\Lambda = \left]-\tfrac{L}{2}, \tfrac{L}{2}\right]^3,$$

with periodic boundary conditions, and described by a many-body self-adjoint Hamiltonian of the form

$$H_N = -\frac{1}{2}\sum_{i=1}^{N}\Delta_{x_i} + \sum_{1\le i<j\le N} v(x_i - x_j),$$

acting on the Hilbert space of symmetric wave functions $L^2_s(\Lambda^N)$ Another point of view consists in adopting the bosonic Fock space formalism in the momentum representation. Defining the Fourier transform of the potential $\hat{v}(\mathbf{k})$, one can implement the many-body Hamiltonian

$$\mathcal{H}_N = \frac{1}{2}\sum_{\mathbf{p}\in\Xi_L}|\mathbf{p}|^2 + \frac{1}{2L^3}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}\in\Xi_L^{>}}\hat{v}(\mathbf{k})a^*_{\mathbf{p}+\mathbf{k}}a^*_{\mathbf{q}-\mathbf{k}}a_{\mathbf{p}}a_{\mathbf{q}}, \tag{1.1}$$

which acts on the Fock space

$$\Gamma_s(\ell^2(\Xi_L)) = \bigoplus_{n=0}^{+\infty}\Gamma^n_s(\ell^2(\Xi_L)),$$

where $\Xi_L = \frac{2\pi\mathbb{Z}^3}{L}$ and $\Gamma^n_s(\ell^2(\Xi_L)) := \otimes^n_s\ell^2(\Xi_L)$ denotes the symmetric n–fold tensor product of $\ell^2(\Xi_L)$, with the convention $\otimes^0_s\ell^2(\Xi_L) = \mathbb{C}$. If we denote by P_N the projector onto the N–particle subspace of $\Gamma_s(\ell^2(\Xi_L))$, it is possible to see that $P_N\mathcal{H}_NP_N$ and H_N are unitarily equivalent.

Within these frameworks, it is possible to employ variational estimates to obtain information about the bottom of H_N spectrum. This program has been carried out by changing the scaling of the interaction's parameters and of the volume with the number of particles N. Two widely employed scaling regimes are given by the Gross–Pitaevskii [7, 14] and the mean-field approximations [20, 21, 34]. Restricting to these regimes, it was proved that the gas exhibits *Bose-Einstein condensation*. However, the standard quantum-mechanical approach is not well adapted to the treatment of bosonic theories in the infinite-volume limit, nor to the investigation of spectral properties at positive temperature.

This article reviews two recent applications [37, 19] of C^* and W^* algebraic techniques [2, 11, 12, 13, 17] to the study of homogeneous Bose gases, presented at IQSA2025. These approaches address structural and dynamical questions that lie beyond the reach of the standard N-particle quantum-mechanical framework.

The first novelty of the algebraic approach is the possibility of working directly in the infinite volume limit. This requires to generalize the notion of *Gibbs states* – the equilibrium states for finite volume systems – to models in the infinite volume $\mathbb{R}^3$ and having an infinite number of particles. This generalization is given by the renowned Kubo-Martin-Schwinger (KMS) states [11, Definition 5.3.1]. For quantum systems, the time evolution is introduced by means of a strongly continuous one-parameter group of *-automorphism $t \to \tau_t$. *States* of the system are described by linear, positive, normalized, functionals $\omega\colon \mathfrak{A} \to \mathbb{C}$. Once the dynamics and an inverse temperature β are fixed, a state ω is said to satisfy the (τ,β)-KMS condition for the C^*-dynamical system $(\mathfrak{A},\tau)$ if there exists a strongly dense *-subalgebra $\mathfrak{A}_\tau$ of $\mathfrak{A}$, contained in the set of the τ-analytic elements, for which the following identity is verified

$$\omega(\mathfrak{a}\tau_{i\beta h}(\mathfrak{b})) = \omega(\mathfrak{b}\mathfrak{a}), \quad \mathfrak{a},\mathfrak{b}\in\mathfrak{A}_\tau. \tag{1.2}$$

Moreover, the above framework admits a natural reformulation at the level of W^*–algebras, replacing strongly continuous one-parameter groups with σ-weakly continuous automorphism groups and states with normal states. A classical analogue of the KMS condition was introduced in [25]. One of the novel contributions of [37] is the formulation of a weaker version of this classical KMS condition, specifically tailored to the bosonic setting. Within this framework, it can be shown—at least in the case of the free

Bose gas—that infinite-volume quantum KMS states for the quantum C^*–algebra converge, in a suitable high-density limit, to states on the classical algebra satisfying the classical KMS condition.

A second approach consists in considering a sequence of finite-volume systems in which the density $n := \frac{N}{L^3}$ is kept fixed. For each N, one then works with the W^*–algebra of bounded operators $\mathcal{B}(L^2_s(\Lambda^N))$. In this setting, one can construct one–quasiparticle vectors in the standard representation associated with the KMS state of the Bogoliubov theory. These vectors are obtained by acting with *left* and *right* creation operators, see Eqs. (3.4)–(3.5), on the KMS vector.

A key advantage of this approach is that finite-temperature effects are incorporated directly into the Liouvillean [17]

$$L^L_\nu = L^L_{\mathrm{Bog},\nu} + \sqrt{\kappa}L^L_{3,2} + \kappa L^L_4,$$

which replaces the Hamiltonian in the Araki-Woods representation.

Taken together, these results illustrate the versatility of the algebraic approach: it provides both a natural language for describing semiclassical limits and an efficient tool for finite-temperature perturbation theory. The unifying theme is that equilibrium states are best understood at the level of operator algebras, where density scaling, condensate structure, and dynamical effects can be analyzed in a coherent and mathematically controlled way.

This review is organized as follows. In Section 2 we introduce a suitable C^*-algebraic framework for the semiclassical analysis of the condensate. In Subsection 2.1 we present the formalism developed in [37], while Subsection 2.2 is devoted to the discussion of the main result, Theorem 2.7, which establishes the connection between quantum and classical equilibrium states. Finally, Section 3 introduces the framework of [19] and discusses an application to the computation of damping effects.

2 C^*–algebraic frameworks

In this section we will review the framework adopted in [37] to describe the semiclassical analysis of equilibrium states for a non-interacting Bose gas.

2.1 Construction of the algebras

We denote by $h \in [0,+\infty[$ the semiclassical parameter of the theory. As will become clear below, this parameter can be related to the density of the gas. Let

$$E := L^2(\mathbb{R}^3, \mathrm{d}\mathbf{x})$$

be the Hilbert space of square-integrable functions on $\mathbb{R}^3$. This space carries a natural non-degenerate symplectic form defined by the imaginary part of the scalar product,

$$\sigma\colon E\times E\to\mathbb{R}, \qquad \sigma(f,g) = \mathrm{Im}\langle f,g\rangle. \tag{2.1}$$

For every $h \in [0,+\infty[$, we define the unital *Weyl* *-algebra

$$\Delta(E, h\sigma) := \mathrm{span}_{\mathbb{C}}\{W^h(f) \ : \ f\in E\},$$

obtained as the quotient of the free *-algebra generated by the Weyl elements $W^h(f)$ with respect to the canonical commutation relations

$$\begin{aligned}W^h(f)W^h(g) &= e^{-\frac{ih}{2}\sigma(f,g)}W^h(f+g),\\ (W^h(f))^* &= W^h(-f),\\ W^h(0) &= \mathbb{I}.\end{aligned}$$

We note that for $h=0$, the algebra becomes commutative and hence, it can be mapped to an algebra of functions. In particular, the algebra $\mathcal{W}(E,0)$ describes a classical theory.

The above *–algebra can be completed to a C^*-algebra via a suitable C^*–norm which can be constructed as

$$\|A\| = \sup_{\omega\in S(\Delta(E,h\sigma))} \omega(A^*A),$$

where $S(\Delta(E,h\sigma))$ is the state space of the Weyl *-algebra. We will denote this completion by $\mathcal{W}(E,h\sigma) := \overline{\Delta(E,h\sigma)}^{\|\cdot\|}$. See [6, 11, 37] for a more detailed analysis of the construction of Weyl algebras.

REMARK 2.1: (**Restriction to subspaces**) The above construction also hold if E is taken to be a real normed vector space with an arbitrary, non-degenerate symplectic form σ. In particular, we can restrict to the Schwartz functions space $E_0 = \mathcal{S}(\mathbb{R}^3)$. Then, $\mathcal{W}(E_0,h\sigma)$ is C^*–subalgebra of $\mathcal{W}(E,\sigma)$ with respect to the same C^*-norm.

◇

Dynamics and equilibrium for the quantum and classical algebras. The non-interacting dynamics on $\mathcal{W}(E,h\sigma)$ is usually introduced by means of a one-parameter group of *-automorphisms $t\to\tau_{h,t}$ acting on the Weyl elements as

$$\tau_{h,t}(W^h(f)) = W^h(e^{iHt}f),$$

where H is some self-adjoint operator acting on the one-particle space E.

REMARK 2.2: (**Interpretation of the dynamics**) The motivation for the introduction of the latter dynamics is semiclassical in nature. E is a space of test functions for the effective wavefunctions of the single particle. The dynamics on the quantum Weyl algebra is introduced via an automorphism, whose action is implemented on the space E. In the $h\to 0^+$ limit, the interpretation of the latter space does not change, i.e. the theory is still fundamentally quantum and only the resulting macroscopic setting is classical. In particular, we expect that the semi-classical limit of the dynamics $\tau_{h,t}(W^h(f)) = W^h(e^{iHt}f)$ should be $W^0(e^{iHt}f) = \tau_{0,t}(W^0(f))$.

◇

In the case of the classical algebra $W^h(E,0)$, we need to introduce some additional structures

Poisson brackets. Following [6], we equip $\mathcal{W}(E,0)$ with a Poisson structure. We take the dense *-algebra $\Delta(E,0) \subset \mathcal{W}(E,0)$ as the domain of the Poisson bracket. These are defined on Weyl elements as

$$\{W^0(f), W^0(g)\} := \sigma(g,f) W^0(f+g) \in \Delta(E,0), \tag{2.2}$$

and extended on $\Delta(E,0)$ by linearity.

Infinitesimal generators. Classical infinitesimal generators are introduced as field functions

$$\Phi_0(f)[g] := \mathrm{Re}\langle f|g\rangle, \qquad f \in E,$$

whose exponential define the classical Weyl elements $W^0(f) = e^{i\Phi_0(f)}$. These generators can be further decomposed in terms of the *classical creation/annihilation operators*

$$a_0^*(f) := \frac{\Phi(f) - i\Phi_0(if)}{\sqrt{2}}, \qquad a_0(f) := \frac{\Phi(f) + i\Phi_0(if)}{\sqrt{2}}. \tag{2.3}$$

Classical states on the commutative Weyl algebra $\mathcal{W}(E,0)$ can be extended to field operators, provided they are *analytic*, meaning that their GNS representation is regular and the cyclic vector is analytic for the associated field generators.

A distinguished class of analytic states is given by quasi-free states, characterized by Gaussian expectations

$$\omega_s(W^0(f)) = \exp\left(-\tfrac{1}{4}s(f,f)\right),$$

with s a real symmetric bilinear form. Analyticity ensures that expectation values of products of field operators can be computed by differentiating the Weyl expectation values.

REMARK 2.3: More generally, one can introduce C^{2k} states, for which moments of field operators up to order k are well defined. This regularity property plays an important role in the formulation of the weak KMS condition for commutative Weyl C^*-algebras.

◇

KMS condition for the quantum algebra. We fix $h > 0$ in this paragraph. We have introduced the quantum (τ_h, β)–KMS condition in the Introduction (see Eq. (1.2)). This definition requires the *-automorphism group to be strongly continuous, in order to admit an analytic continuation to imaginary times $t \mapsto i\beta h$.

In the case of a free gas, the *-automorphism group is constructed from the one-particle Hamiltonian $H = -\frac{1}{2}\Delta$, which represents the kinetic energy in units $\hbar = m = 1$[1]. Formally, the dynamics acts on Weyl elements by

$$\tau_{h,t}(W^h(f)) = W^h(e^{iHt}f).$$

[1] Here H is understood as the unique self-adjoint extension of the Laplacian initially defined on the Schwartz space.

However, this automorphism group is not strongly continuous. Indeed, for every $f \in E$ that is not invariant under e^{iHt}, one has

$$\|\tau_{h,t}(W^h(f)) - W^h(f)\| = \|W^h(e^{iHt}f) - W^h(f)\| = 2,$$

showing that continuity at $t = 0$ fails in the C^*-norm.

Nevertheless, it is still possible to identify the appropriate class of equilibrium states for the algebra $\mathcal{W}(E_0, h\sigma)$. We denote this family by ω_h^ρ, where ρ represents the total particle density. These states, together with their GNS representations $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$, were first constructed by Araki and Woods [2]. Subsequently, Cannon [13] recovered the same states as thermodynamic limits of the corresponding finite-volume Gibbs states.

It can be verified that the states ω_h^ρ satisfy a W^*–KMS condition at inverse temperature β, in their own GNS representation. An important outcome of this derivation is that, at sufficiently low temperatures $\frac{1}{\beta}$, the same dynamics admits a whole family of equilibrium states. More precisely, define the critical density

$$\rho_c(\beta h) := \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{e^{-\beta h \frac{|\mathbf{p}|^2}{2}}}{1 - e^{-\beta h \frac{|\mathbf{p}|^2}{2}}}.$$

Then, for every $\rho > \rho_c(\beta h)$, the corresponding equilibrium states act on the Weyl elements as

$$\omega_h^\rho(W^h(f)) = \exp\left(-\frac{h}{4}\left\langle f, (\mathbb{I} + e^{-\beta hH})(\mathbb{I} - e^{-\beta hH})^{-1} f\right\rangle\right) \times \exp\left(-2h(\rho - \rho_c(\beta h))\left|\int d\mathbf{x}\, f(\mathbf{x})\right|^2\right). \quad (2.4)$$

Thus, for densities $\rho > \rho_c(\beta h)$, one obtains a continuum of equilibrium states at the same temperature. States corresponding to different values of ρ give rise to mutually disjoint GNS representations. The factor in the second line of Eq. (2.4) keeps track of the macroscopic occupation of the ground state of the Bose gas. The fraction of particles occupying this state is given by $\rho - \rho_c(\beta h)$, whereas $\rho_c(\beta h)$ counts the fraction of particles belonging to the thermal background of the gas.

KMS condition for the classical algebra. The introduction of the classical KMS condition in[25] is motivated by a formal $h \to 0^+$ limit of the quantum condition. We repeat this argument in the specific setting of Weyl algebras. The quantum KMS condition (1.2) can be equivalently written as

$$\omega_h\left(W^h(f)\frac{1}{ih}\left(\tau_{i\beta h}(W^h(g)) - W^h(g)\right)\right) = \omega_h\left(\frac{1}{ih}[W^h(g), W^h(f)]\right). \quad (2.5)$$

In the semi-classical limit $h \to 0^+$, the right-hand side of (2.5) should converge to $\omega_0\big(\{W^0(f), W^0(g)\}\big)$, while the left-hand side should converge to $\beta\omega_0\big(W^0(f)\,\delta_0(W^0(g))\big)$, where δ_0 is the derivation generated by the *-automorphism group τ_0. Since in general, for Weyl algebras the *–automorphism cannot be taken to be strongly continuous, we cannot take δ_0 to be a true derivation on the C^*–algebra. However, it is possible to enlarge the class of derivations by defining **weak–derivations**. In [37], these are completely characterized in terms of linearity and continuity properties. At the end, one concludes that the correct choice for δ_0 is given by

$$\delta_0 \colon \Delta(E_0,0) \to C(E), \quad \delta_0(W^0(f)) := i\Phi_0(iHf)W^0(f),\ f \in E_0, \tag{2.6}$$

where H is the self-adjoint operator appearing in the definition of τ_0. In (2.6), $C(E)$ is the space of continuous, not necessarily bounded, functions from E to $\mathbb{C}$.

This specific form for the derivation allows us to introduce a classical KMS condition for the Weyl C^*-algebra $\mathcal{W}(E,0)$. Following the intuition obtained from Eq. (2.5) we define

DEFINITION 2.4: (**Weak classical KMS condition**) Let E be a normed symplectic space, $\beta \in \mathbb{R}$, and δ_0 a weak derivation satisfying $\delta_0(W^0(f)) := i\Phi_0(iHf)W^0(f)$ for all $f \in E_0$. Then, if $\omega_0 \in \mathcal{S}(\mathcal{W}(E,0))$ is a C^2 state as in Rmk. 2.3, we say that ω_0 satisfies the weak (δ_0,β)-KMS property if

$$\omega_0(\{a,b\}) = \beta\,\omega_0(b\delta_0(a)), \quad \forall\, a,b \in \Delta(E_0,0), \tag{2.7}$$

where the Poisson bracket has been defined in (2.2).

◇

Note that in 2.4 we cannot require less than C^2-regularity on the classical state ω_0, since we need to compute the expectation value of $\Phi_0(f)$, for $f \in E$; see the discussion on regularity in Rmk. 2.3. The foregoing definition is suitable for Weyl C^*-algebras since it does not involve continuity properties of time evolution. We remark that one could also give a definition more in line with the approach of W^*-dynamical systems and verify that it is satisfied by weak KMS states, see [37, Appendix C]

2.2 Semiclassical analysis of the condensate

In this section, we show how classical states on the Weyl algebra $\mathcal{W}(E_0,0)$ satisfying the classical weak (δ_0,β)–KMS condition can be constructed as semiclassical limits of the quantum equilibrium states ω_h^ρ. The condensate will persists in the semiclassical limit, thereby providing an effective description of the quantum Bose gas in the regime of large densities.

Berezin quantization map. We begin by introducing a quantization map linking the classical and quantum Weyl algebras. It can be seen as a generalization of Berezin quantization map [5] to the Weyl C^*–algebra.

DEFINITION 2.5 (**Abstract quantization map**): We define a net of linear maps

$$\big(Q_h \colon \Delta(E,0) \to \Delta(E,h\sigma)\big)_{h\in[0,+\infty[} \tag{2.8}$$

by extending linearly the action on Weyl elements given by

$$Q_h\big(W^0(f)\big) := e^{-\frac{h}{4}\|f\|^2}\,W^h(f), \qquad h \in [0,+\infty[,\ f \in E. \tag{2.9}$$

◇

This map allows us to pass from classical to quantum observables between the corresponding Weyl algebras. We would like to further define the pull-back action of Q_h on quantum states ω_h, obtaining a net of classical states

$$(\omega_h \circ Q_h)_{h\in[0,+\infty[}.$$

Further, it would be desirable to establish an appropriate notion of continuity for this quantization procedure, ensuring that the limit $h \to 0^+$ is well defined. These properties are established in the next proposition, whoose proof can be found in [37]

PROPOSITION 2.6: The abstract quantization map defined on (2.9) satisfies the following.

(i) $\|Q_h(c)\|_h \le \|c\|_0$ for every $c \in \Delta(E,0)$;

(ii) for every $c \in \Delta(E,0)$, $c \ge 0$ implies that $Q_h(c) \ge 0$ as elements of the respective C^*-algebras $\mathcal{W}(E,0)$ and $\mathcal{W}(E,h\sigma)$.

As a consequence, Q_h on $\Delta(E,0)$ can be extended to a positive map

$$Q_h \colon \mathcal{W}(E,0) \to \mathcal{W}(E,h\sigma). \tag{2.10}$$

◇

Properties *(i)* and *(ii)* are the necessary continuity and positivity properties we wanted to ensure that $\omega_h \circ Q_h$ is a well defined algebraic state having a well defined weak*–limit points as $h \to 0^+$.

Limit of the condensate. With the quantization map Q_h at our disposal it is now possible to state the main theorem of this section

THEOREM 2.7: Let the dynamics of the Weyl algebra $\mathcal{W}(E_0,h\sigma)$ be implemented by the *-automorphism $\tau_{h,t}(W^h(f)) = W^h(e^{iHt}f)$ for all $f \in E_0$, where $H = -\frac{\Delta}{2}$. Then, if $\overline{\rho}(h) > \rho_c(\beta h)$ for all $h > 0$ and $h(\overline{\rho}(h) - \rho_c(\beta h)) \xrightarrow{h\to 0^+} \alpha \ge 0$, the net of quantum $(\tau_h,\beta h)$-KMS states $\omega_h^{\overline{\rho}(h)}$ with

$$\omega_h^{\overline{\rho}(h)}(W^h(f)) = \exp\left\{-\frac{h}{4}\left(\langle f,(I+e^{-\beta hH})(I-e^{-\beta hH})^{-1}f\rangle\right\}\right.$$
$$\left.\times \exp\left\{2^\nu(\overline{\rho}(h)-\rho_c(\beta h))|\int_{\mathbb{R}^3}\mathrm{d}^\nu x f(x)|^2\right)\right\} \tag{2.11}$$

converges for $h \to 0^+$ to the classical state

$$\omega_0^\alpha(W^0(f)) = \exp\left\{-\frac{1}{2}\left(\langle f(\beta H)^{-1}f\rangle + 2^\nu\alpha|\int_{\mathbb{R}^\nu}\mathrm{d}^\nu x f(x)|^2\right)\right\} \tag{2.12}$$

in the sense that

$$\omega_h^{\overline{\rho}(h)} \circ Q_h(c) \to \omega_0^\alpha(c), \quad \text{for all } c \in \mathcal{W}(E_0,0). \tag{2.13}$$

The weak*-limit points ω_0^α are labeled by the parameter $\alpha \ge 0$ and they all satisfy the (δ_0,β)-weak KMS condition.

$\diamond$

A few comments are in order:

- The condition $\overline{\rho}(h) > \rho_c(\beta h)$ is necessary in order to preserve the condensate for every value of the semiclassical parameter $h > 0$. As discussed above, the parameter $h \in [0, +\infty[$ is not merely a formal semiclassical parameter: it is directly linked to the density scale of the system. In fact, it can be expressed in terms of the critical density as

$$h = h_0 \left(\frac{\rho_c(\beta h_0)}{\rho_c(\beta h)} \right)^{\frac{2}{3}}, \tag{2.14}$$

  where h_0 is a fixed reference value.

  In particular, as $h \to 0^+$ one has $\rho_c(\beta h) \to +\infty$ Hence, the semiclassical limit $h \to 0^+$ corresponds to a regime in which the critical density diverges. Consequently, for every fixed total density $\overline{\rho} > 0$, there exists a threshold value $h_1 > 0$ such that for $h < h_1$ one has $\overline{\rho} < \rho_c(\beta h)$. In this regime, Bose–Einstein condensation cannot occur, and the condensate state $\omega_h^{\overline{\rho}}$ is no longer defined. Therefore, the semiclassical limit of $\omega_h^{\overline{\rho}} \circ Q_h$ cannot be taken at fixed density: the density must scale appropriately with h in order to remain in the condensed phase.

- As the semi-classical parameter h is dimensionless, α carries the dimensions of an inverse volume and can be interpreted as a *renormalized density*. Indeed, for the *single excitation* number operator $N_0(f) := a_0(f)^* a_0(f)$ one finds

$$\omega_0^{\alpha}(N_0(f)) = \langle f | (\beta H)^{-1} f \rangle + 2^{\nu} \alpha \langle f | 1 \rangle \langle 1 | f \rangle \tag{2.15}$$

  which decomposes as the sum of two quadratic forms: the first accounts for the contribution of the classical background, while the second quantifies the condensate fraction.

3 Perturbation theory for an interacting Bose gas

In this section we will briefly discuss another recent success of the algebraic formalism in the analysis of finite-temperature Bose gases. The results of this section review the original paper [19].

We employ the Araki–Woods representation to construct a rigorous and systematic perturbation theory for an interacting bosonic gas. Set $\Xi_L^{>} := \Xi_L \setminus \{\mathbf{0}\}$. We will work with the W^* algebra of bounded operators $\mathcal{B}(\Gamma_s(\ell^2(\Xi_L^{>})))$. Within this framework, it is possible to compute the imaginary correction to the dispersion relation of Bogoliubov quasiparticles. This amounts to the Fermi Golden rule for a three body term. For one-quasiparticle states, this correction is the sum of two terms:

$$-\gamma_{\mathrm{B}}(\mathbf{k}, \beta, \nu) - \gamma_{\mathrm{L}}(\mathbf{k}, \beta, \nu).$$

In the literature the term γ_{B} goes under the name of the *Beliaev damping* [4, 18, 26, 27, 36] . This correction persists down to the zero temperature. The term γ_{L} is called the *Landau damping* [27, 36, 38].

This term is absent at zero temperature. Experimental results [15, 16, 24] seem to validate theoretical predictions for the two coefficients.

We define an effective bosonic Hamiltonian acting on the Fock space $\Gamma_s(\ell^2(\Xi_L^>))$ by

$$
\begin{aligned}
H_\nu^L &= H_{\mathrm{Bog},\nu}^L + \sqrt{\kappa} H_{3,\nu}^L + \kappa H_4^L, \qquad (3.1)\\
H_{\mathrm{Bog},\nu}^L &:= \sum_{\mathbf{p}\in\Xi_L^>} \Big(\frac{\mathbf{p}^2}{2} + \frac{\nu\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})}\Big) a_{\mathbf{p}}^* a_{\mathbf{p}} \\
&\quad + \sum_{\mathbf{p}\in\Xi_L^>} \Big(\frac{\nu\hat{v}(\mathbf{p})}{2\hat{v}(\mathbf{0})} a_{\mathbf{p}} a_{-\mathbf{p}} + \mathrm{h.c.}\Big), \\
H_{3,\nu}^L &:= \frac{1}{L^{d/2}} \sum_{\mathbf{k},\mathbf{p},\,\mathbf{p}+\mathbf{k}\in\Xi_L^>} \frac{\sqrt{\nu}\,\hat{v}(\mathbf{k})}{\sqrt{\hat{v}(\mathbf{0})}} \Big(a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{k}} a_{\mathbf{p}} + \mathrm{h.c.}\Big), \\
H_4^L &:= \frac{1}{2L^d} \sum_{\substack{\mathbf{p},\mathbf{q},\mathbf{k} \\ \mathbf{p}+\mathbf{k},\,\mathbf{q}-\mathbf{k}\in\Xi_L^>}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}.
\end{aligned}
$$

The introduction of this Hamiltonian can be justified starting from the first-principles model (1). The key step consists in isolating the zero-mode creation and annihilation operators $a_{\mathbf{0}}^*, a_{\mathbf{0}}$, corresponding to particles with momentum $\mathbf{k} = \mathbf{0}$. In position representation, this mode is associated with the constant function $\otimes^N|1\rangle \in L_s^2(\Lambda^N)$, i.e. the ground state of the non-interacting model.

One then performs a formal substitution of the zero-mode operators by the square root of the zero-mode number operator,

$$a_{\mathbf{0}},\, a_{\mathbf{0}}^* \;\longrightarrow\; \sqrt{N_0} \simeq \mathbb{I}\sqrt{N},$$

where N_0 denotes the occupation number of the condensate. Within this approximation, the natural definition of the effective parameter ν is

$$\nu := \frac{N\hat{v}(\mathbf{0})}{L^3},$$

which is kept fixed in the joint limit $L, N \to \infty$.

We emphasize that this substitution can be rigorously justified only at first order in perturbation theory[33, 35]. The analysis of higher-order contributions requires a more careful treatment of the zero-mode operators. Nevertheless, the above replacement is sufficient for the computation of the imaginary part of the energy shift arising from the Fermi golden rule.

The parameter κ appearing in (3.1) is an artificial *small* constant, which keeps track of the correct order in perturbation theory. The Hamiltonian H_{bg}^L is usually called the *Bogoliubov Hamiltonian* and it will be treated as the main part of the full Hamiltonian. Being quadratic in $a_{\mathbf{k}}$, $a_{\mathbf{k}}^*$, it can be exactly diagonalized by a *Bogoliubov transformation* [9]

$$a_{\mathbf{k}}^* = c_{\mathbf{k}} b_{\mathbf{k}}^* - s_{\mathbf{k}} b_{-\mathbf{k}} \qquad a_{\mathbf{k}} = c_{\mathbf{k}} b_{\mathbf{k}} - s_{\mathbf{k}} b_{-\mathbf{k}}^*, \qquad (3.2)$$

where $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^*$ are the *Bogoliubov quasiparticle* annihilation and creation operators, whereas $c_{\mathbf{k}}$ and $s_{\mathbf{k}}$ are suitable coefficients satisfying $c_{\mathbf{k}}^2 - s_{\mathbf{k}}^2 = 1$. These transformations can always be implemented by a unitary operator U on $\Gamma_S(\ell^2(\Xi_L^>))$ under relatively mild hypothesis on $\hat{v}$ [19]. After this diagonalization, the Bogoliubov Hamiltonian becomes

$$UH_{\text{bg}}^L U^* = \sum_{\mathbf{p}\in\Xi_L^>} b_{\mathbf{p}}^* b_{\mathbf{p}} \omega_{\text{bg}}(\mathbf{p}), \qquad \omega_{\text{bg}}(\mathbf{p}) = \sqrt{\frac{|\mathbf{p}|^4}{4} + \nu\hat{v}(\mathbf{p})|\mathbf{p}|^2}.$$

Let us now fix a positive temperature $\frac{1}{\beta} \in]0, T_c[$. Here $T_c > 0$ denotes a (generally unknown) value such that for $\frac{1}{\beta} < T_c$ one expects the presence of Bose–Einstein condensation, i.e. a macroscopic occupation of the ground state.

For the quadratic Bogoliubov Hamiltonian, the unique KMS state is given by the Gibbs functional

$$\omega_\beta(A) = \frac{\text{Tr}\Big(Ae^{-\beta H_{\text{Bog}}^L}\Big)}{\text{Tr}\Big(e^{-\beta H_{\text{Bog}}^L}\Big)}, \qquad A \in \Gamma_s\big(\ell^2(\Xi_L^>)\big). \tag{3.3}$$

This thermal state can be realized as a Fock vacuum state in a doubled representation, namely on the Hilbert space

$$\Gamma_s\big(\ell^2(\Xi_L^>) \oplus \ell^2(\Xi_L^>)\big).$$

More precisely, one introduces left and right representations $\pi_{\beta,\text{l}}$ and $\pi_{\beta,\text{r}}$ of the creation and annihilation operators, defined by

$$\begin{aligned}
b_{\beta,\text{l}}^*(\mathbf{k}) &:= \pi_{\beta,\text{l}}(b^*(\mathbf{k})) = \big(1 - e^{-\beta\omega_{\text{Bog}}(\mathbf{k})}\big)^{-\frac{1}{2}} b_{\text{l}}^*(\mathbf{k}) + \big(e^{\beta\omega_{\text{Bog}}(\mathbf{k})} - 1\big)^{-\frac{1}{2}} b_{\text{r}}(\mathbf{k}),\\
b_{\beta,\text{l}}(\mathbf{k}) &:= \pi_{\beta,\text{l}}(b(\mathbf{k})) = \big(1 - e^{-\beta\omega_{\text{Bog}}(\mathbf{k})}\big)^{-\frac{1}{2}} b_{\text{l}}(\mathbf{k}) + \big(e^{\beta\omega_{\text{Bog}}(\mathbf{k})} - 1\big)^{-\frac{1}{2}} b_{\text{r}}^*(\mathbf{k}),\\
b_{\beta,\text{r}}^*(\mathbf{k}) &:= \pi_{\beta,\text{r}}(b^*(\mathbf{k})) = \big(e^{\beta\omega_{\text{Bog}}(\mathbf{k})} - 1\big)^{-\frac{1}{2}} b_{\text{l}}(\mathbf{k}) + \big(1 - e^{-\beta\omega_{\text{Bog}}(\mathbf{k})}\big)^{-\frac{1}{2}} b_{\text{r}}^*(\mathbf{k}),\\
b_{\beta,\text{r}}(\mathbf{k}) &:= \pi_{\beta,\text{r}}(b(\mathbf{k})) = \big(e^{\beta\omega_{\text{Bog}}(\mathbf{k})} - 1\big)^{-\frac{1}{2}} b_{\text{l}}^*(\mathbf{k}) + \big(1 - e^{-\beta\omega_{\text{Bog}}(\mathbf{k})}\big)^{-\frac{1}{2}} b_{\text{r}}(\mathbf{k}).
\end{aligned}$$

Here the operators $b_{\text{l/r}}(\mathbf{k})$ and $b_{\text{l/r}}^*(\mathbf{k})$ act respectively on the left and right components of $\Gamma_s\big(\ell^2(\Xi_L^>) \oplus \ell^2(\Xi_L^>)\big)$, and satisfy the canonical commutation relations

$$[b_{\text{l/r}}(\mathbf{k}), b_{\text{l/r}}(\mathbf{q})] = [b_{\text{l/r}}^*(\mathbf{k}), b_{\text{l/r}}^*(\mathbf{q})] = 0, \tag{3.4}$$

$$[b_{\text{l/r}}(\mathbf{k}), b_{\text{l/r}}^*(\mathbf{q})] = \delta_{\mathbf{k},\mathbf{q}}, \qquad b_{\text{l/r}}(\mathbf{k})\Omega = 0, \tag{3.5}$$

where Ω denotes the Fock vacuum of the doubled space.

In this representation, all thermal effects are encoded in the modified creation and annihilation operators and the dynamics is generated by the Liouvillean

$$L_\nu^L = \pi_{\beta,\text{l}}(H_\nu^L) - \pi_{\beta,\text{r}}(H_\nu^L) = L_{\text{bg},\nu}^L + \sqrt{\kappa} L_{3,\nu}^L + \kappa L_4^L.$$

Perturbative computations can be performed using excited vectors of the form

$$b_{\text{l}}^*(\mathbf{k}_1)\cdots b_{\text{l}}^*(\mathbf{k}_n) b_{\text{l}}^*(\mathbf{p}_1)\cdots b_{\text{l}}^*(\mathbf{p}_m)\,\Omega,$$

and, since Ω is a genuine Fock vacuum, Wick's theorem can be applied directly to compute matrix elements. In particular, by computing the Fermi Golden Rule correction generated by the perturbation $\sqrt{\kappa}L^L_{3,\nu}$ around the unperturbed one-particle state $b_1^*(\mathbf{k})\Omega$, we obtain the following expression for the corresponding damping rate coefficients:

THEOREM 3.1: In the thermodynamic limit $L \to \infty$, the following estimates hold:

1. For small momenta and temperature/momentum ratios we have

$$\gamma_{\mathrm{B}}(\mathbf{k},\beta,\nu) = \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{640\pi}\frac{|\mathbf{k}|^5}{\nu^{5/2}}\left(1+O\Big(\frac{1}{(\beta\sqrt{\nu}|\mathbf{k}|)^3}\Big)+O\Big(\frac{|\mathbf{k}|^2}{\nu}\Big)\right) \quad \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}},\ \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \to 0. \tag{3.6}$$

2. For small momenta, temperatures and the momentum/temperature ratios we have

$$\gamma_{\mathrm{L}}(\mathbf{k};\beta,\nu) = \frac{3\pi^3\hat{v}(\mathbf{0})\nu^{3/2}}{40(\beta\nu)^4}\frac{|\mathbf{k}|}{\sqrt{\nu}}\left(1+O\left(\beta\sqrt{\nu}|\mathbf{k}|\right)+O\Big(\frac{1}{(\beta\nu)^2}\Big)\right) \quad \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}},\ \frac{1}{\beta\nu},\ \beta\sqrt{\nu}|\mathbf{k}| \to 0. \tag{3.7}$$

◇

4 Conclusions

The standard N-body formalism of quantum mechanics and the algebraic approach provide complementary perspectives on the properties of a Bose gas. In this review, we have presented two applications that contribute to enhancing our understanding of bosonic systems in two distinct directions. Nevertheless, several important questions remain open.

A natural continuation of the results in [37], discussed in Section 2, would be their extension to interacting systems. Such a development would likely require combining the techniques of [12, 19, 37] with new conceptual and technical ideas. On the perturbative side, although we have derived damping rates for bosonic excitations, the mathematical foundations of the perturbative expansion are far from being fully understood. In particular, it is natural to ask whether a convergent perturbation series can be established, and how the present weak-potential expansion relates to the more commonly studied low-density regime.

We expect that further developments of the algebraic approach will provide useful insights into these questions and contribute to a deeper structural understanding of finite-temperature Bose gases.

Acknowledgements. L. P. is grateful for the support of the National Group of Mathematical Physics (GNFM-INdAM).

Data availability statement. Data sharing is not applicable to this article as no new data were created or analysed in this study.

Conflict of interest statement. The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter discussed in this manuscript.

References

[1] Abo-Shaeer, J. R., Ketterle, W., Raman, C., Vogels, J. M., Xu, K., *Experimental Observation of the Bogoliubov Transformation for a Bose–Einstein Condensed Gas*. Phys. Rev. Lett. **88**, 6, 060402 (2002).

[2] Araki, H., Woods, E. J., *Representations of the Canonical Commutation Relations Describing a Nonrelativistic Infinite Free Bose Gas. J. Math. Phys.* **4**, 637-662 (1963)

[3] Beauvois, K., Dawidowski, J., Godfrin, H., Krotscheck, E., Fåk, B., Ollivier, J., Sultan, A., *Dispersion Relation of Landau Elementary Excitations and Thermodynamic Properties of Superfluid* ^{4}He. Phys. Rev. B **103**, 10, 104516 (2021).

[4] Beliaev, S. T., *Energy Spectrum of a Non-Ideal Bose Gas*. Sov. Phys. JETP **34**, 7 , 299 (1958) (translation of Zh. Eksp. Teor. Fiz. **34**, 433 (1958)).

[5] Berezin F. A. *Quantization.* Math. USSR Izv. **8** 1109–1163. (1974)

[6] Binz E., Honegger R. & Rieckers A. *Construction and uniqueness of the* C^**-Weyl algebra over a general pre-symplectic space. J. Math. Phys.* **45**, 2885-2907 2004

[7] Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B., *Complete Bose–Einstein Condensation in the Gross–Pitaevskii Regime*. *Comm. Math. Phys.* **359**, 975-1026 (2018)

[8] Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B., *Bogoliubov theory in the Gross–Pitaevskii limit*. *Acta Math.* **222**(2), 219-335 (2019)

[9] Bogolyubov N. N., *On the theory of superfluidity*, J. Phys. (USSR) **11** 23-32 (1947)

[10] Bratteli O., Robinson D. W. *Operator algebras and quantum statistical mechanics I.* Springer-Verlag Berlin Heidelberg (1987).

[11] Bratteli O., Robinson D. W. *Operator algebras and quantum statistical mechanics II.* Springer-Verlag Berlin Heidelberg (1997).

[12] Bucholz D. *Proper condensates. J. Math. Phys.* **63**, 011903 (2022)

[13] Cannon, J. T. *Infinite volume limits of the canonical free Bose gas states on the Weyl algebra. Commun. Math. Phys.* **29**, 89-104 (1973)

[14] Caraci C., Olgiati A. Aubin D. S., Schlein B., *Third Order Corrections to the Ground State Energy of a Bose Gas in the Gross–Pitaevskii Regime*. *Comm. Math. Phys.* **406**, 153 (2025)

[15] Cornell, E. A., Ensher, J. R., Jin, D. S., Matthews, M. R., Wieman, C. E., *Temperature-Dependent Damping and Frequency Shifts in Collective Excitations of a Dilute Bose–Einstein Condensate*. Phys. Rev. Lett. **78**,5, 764 (1997).

[16] Davidson, N., Katz, N., Ozeri, R., Steinhauer, J., *Beliaev Damping of Quasiparticles in a Bose–Einstein Condensate*. Phys. Rev. Lett. **89**, 22, 220401 (2002).

[17] Dereziński J., Jakšić V., Pillet C.-A., *Perturbation Theory of W^*-Dynamics, Liouvilleans and KMS States*. Rev. Math. Phys. **15**, 447–489 (2003).

[18] Dereziński, J., Li, B., Napiórkowski, M., *Beliaev Damping in Bose Gas*. J. Stat. Phys. **191**, 110 (2024).

[19] Dereziński J., Pettinari L., *Damping of phonons in Bose gas at low temperatures*, ArXiv:2602.07701 (2026)

[20] Deuchert, A, Seiringer, R., Yngvason, J. *Bose–Einstein Condensation in a Dilute, Trapped Gas at Positive Temperature* . *Comm. Math. Phys.* **368**, 723-776 (2019)

[21] Dereziński, J., Napiórkowski M., *Excitation spectrum of interacting bosons in the mean-field infinite-volume limit*. *Ann. Herni Poincaré* **15**, 2409-2439 (2014)

[22] Deuchert, A., Seiringer, R. *semi-classical approximation and critical temperature shift for weakly interacting trapped bosons*. *Journal of Functional Analysis* **281**. (2021)

[23] Drago, N., Pettinari, L., van de Ven, C., J., F. *Classical and Quantum KMS states on Spin Lattices Systems. preprint* (2024)

[24] Foot, C. J., Hechenblaikner, G., Hodby, E., Maragò, O. M., *Experimental Observation of Beliaev Coupling in a Bose–Einstein Condensate*. Phys. Rev. Lett. **86**, 2196 (2001).

[25] Gallavotti, G., Verboven, E. *On the classical KMS boundary condition. Nuov Cim B* **28**, 274-186 (1975)

[26] Gavoret, J., Nozières, P., *Structure of the Perturbation Expansion for the Bose Liquid at Zero Temperature*. Ann. Phys. **28**, 3, 349 (1964).

[27] Giorgini, S., *Damping in Dilute Bose Gases: A Mean-Field Approach*. Phys. Rev. A **57**, 2949 (1998).

[28] Giuliani A., Seiringer R., *The Ground State Energy of the Weakly Interacting Bose Gas at High Density*. *J Stat Phys* **135**, 915-934 (2009)

[29] Griffin, A., Hua, S., *Finite-Temperature Excitations in a Dilute Bose-Condensed Gas*. Phys. Rep. **304**, 1–87 (1998).

[30] Hohenberg, P. C., Martin, P. C., *Microscopic Theory of Superfluid Helium*. Ann. Phys. **34**, 2, 291 (1965).

[31] Jakšić, V., Pillet, C.-A.: On a model for quantum friction II: Fermi's golden rule and dynamics at positive temperature, Comm. Math. Phys. **176**, 619 (1996).

[32] Lewin, M., Nam, P. T., Rougerie, N. *Derivation of Hartree's theory for generic mean-field Bose systems*. *Advances in Mathematics*. **254**, 570-621 (2014)

[33] Lieb, E. H., Seiringer, R., Yngvason, J., *Justification of c-Number Substitutions in Bosonic Hamiltonians*. Phys. Rev. Lett. **94**, 080401 (2005).

[34] Lieb E. H., Seiringer R., *proof of Bose-Einstein Condensation for Dilute Trapped Gases*. *Phys. Rev. Lett.* **88**, 170409 (2002)

[35] Lieb, E. H., Solovej, J., P., Seiringer, R., Yngvason, J. *The Mathematics of the Bose Gas and its Condensation*. Birkhäuser Basel (2005)

[36] Liu, W. V., *Theoretical Study of the Damping of Collective Excitations in a Bose–Einstein Condensate*. Phys. Rev. Lett. **79**, 21, 4056 (1997).

[37] Pettinari L., *On Classical Aspects of Bose-Einstein condensation*. Annales Henri Poincaré, (2025).

[38] Pitaevskii, L. P., Stringari, S., *Landau Damping in Dilute Bose Gases*. Phys. Lett. A **235**, 398–402 (1997).