

The damping of phonons in Bose gas at low temperature

Joint work with Jan Dereziński

Lorenzo Pettinari
Itinerant Quantum Math Meetings  - Milan

18 March 2026



Introduction to the physics



Consider an **N-particles, homogeneous Bose gas in finite volume** $]-\frac{L}{2}, \frac{L}{2}]^d$ $d \geq 2$

- For **weakly interacting** gasses, there is a description in terms of quasi-particle excitations with a peculiar **dispersion relation** $\mathbf{p} \rightarrow \omega_{\text{bg}}(\mathbf{p})$



Consider an **N-particles, homogeneous Bose gas in finite volume** $]-\frac{L}{2}, \frac{L}{2}]^d$ $d \geq 2$

- For **weakly interacting** gasses, there is a description in terms of quasi-particle excitations with a peculiar **dispersion relation** $\mathbf{p} \rightarrow \omega_{\text{bg}}(\mathbf{p})$
- Due to interactions, low-energy excitations **decay** and the singular spectrum $\omega_{\text{bg}}(p)$ is **broadened** in the **thermodynamic limit**



Consider an **N-particles, homogeneous Bose gas in finite volume** $]-\frac{L}{2}, \frac{L}{2}]^d$ $d \geq 2$

- For **weakly interacting** gasses, there is a description in terms of quasi-particle excitations with a peculiar **dispersion relation** $\mathbf{p} \rightarrow \omega_{\text{bg}}(\mathbf{p})$
- Due to interactions, low-energy excitations **decay** and the singular spectrum $\omega_{\text{bg}}(p)$ is **broadened** in the **thermodynamic limit**
- **The goal of this talk** is to explain how the W^* -algebraic (**Dereziński–Jaksić–Pillet '03**) formalism can be used to compute decay rates in a weak coupling regime



Consider an **N-particles, homogeneous Bose gas in finite volume** $]-\frac{L}{2}, \frac{L}{2}]^d$ $d \geq 2$

- For **weakly interacting** gasses, there is a description in terms of quasi-particle excitations with a peculiar **dispersion relation** $\mathbf{p} \rightarrow \omega_{\text{bg}}(\mathbf{p})$
- Due to interactions, low-energy excitations **decay** and the singular spectrum $\omega_{\text{bg}}(p)$ is **broadened** in the **thermodynamic limit**
- **The goal of this talk** is to explain how the W^* -algebraic (**Dereziński–Jaksić–Pillet '03**) formalism can be used to compute decay rates in a weak coupling regime
- We do this at a finite temperature, using the **Araki–Woods** representations

Introduction to the model

- **Bose gas:** N bosons in a box with periodic boundary conditions $] -\frac{L}{2}, \frac{L}{2}]^d$
- The **first principle Hamiltonian** is the following

$$H^L = -\frac{1}{2} \sum_{i=1}^N \Delta_i + \frac{1}{2} \sum_{i \neq j} V_L(x_i - x_j)$$

$$H^L : D(H^L) \subset L_s^2 \left(\left] -\frac{L}{2}, \frac{L}{2} \right]^d \right)^N \rightarrow L_s^2 \left(\left] -\frac{L}{2}, \frac{L}{2} \right]^d \right)^N$$

Introduction to the model

- **Bose gas:** N bosons in a box with periodic boundary conditions $] -\frac{L}{2}, \frac{L}{2}]^d$
- The **first principle Hamiltonian** is the following

$$H^L = -\frac{1}{2} \sum_{i=1}^N \Delta_i + \frac{1}{2} \sum_{i \neq j} V_L(x_i - x_j)$$

$$H^L : D(H^L) \subset L_s^2 \left(] -\frac{L}{2}, \frac{L}{2}]^d \right)^N \rightarrow L_s^2 \left(] -\frac{L}{2}, \frac{L}{2}]^d \right)^N$$

- The **density** of the model is given by

$$\frac{N}{L^d} = n(1 + o(N^0)), \quad n > 0$$

Introduction to the model

- **Bose gas:** N bosons in a box with periodic boundary conditions $] -\frac{L}{2}, \frac{L}{2}]^d$
- The **first principle Hamiltonian** is the following

$$H^L = -\frac{1}{2} \sum_{i=1}^N \Delta_i + \frac{1}{2} \sum_{i \neq j} V_L(x_i - x_j)$$

$$H^L : D(H^L) \subset L_s^2 \left(] -\frac{L}{2}, \frac{L}{2}]^d \right)^N \rightarrow L_s^2 \left(] -\frac{L}{2}, \frac{L}{2}]^d \right)^N$$

- The **density** of the model is given by

$$\frac{N}{L^d} = n(1 + o(N^0)), \quad n > 0$$

- V_L is obtained as the periodized version of $V \in L^1(\mathbb{R}^3)$,

$$V_L(x) = \frac{1}{L^d} \sum_{\mathbf{k} \in \Xi_L} e^{i\mathbf{k} \cdot \mathbf{x}} \widehat{V}(\mathbf{k}), \quad \Xi_L = \frac{2\pi\mathbb{Z}^d}{L}$$

Basic hypothesis on the potential



Suppose that

- a) $V \in L^1(\mathbb{R}^d)$ and $V(x) \in \mathbb{R}$;
- b) $\widehat{V} \in L^2(\mathbb{R}^d)$;

Define

$$\nu := \frac{\widehat{V}(\mathbf{0})n}{L^d}$$

and suppose there exists $W \in \mathbb{R}_+$ such that the following hold for all $\nu \in]0, W]$

- c) $\widehat{V}(\mathbf{0}) > 0$ and $\widehat{V}(\mathbf{k}) > -\widehat{V}(\mathbf{0}) \frac{|\mathbf{k}|^2}{2\nu}$;
- d) $V(x) = V(-x)$ which implies $\widehat{V}(\mathbf{k}) = \widehat{V}(-\mathbf{k})$

RMK: We will avoid writing the subscript L on V_L

Improved c-number condensate model



Starting from H_N , we derive an **effective Hamiltonian** on $\Gamma_s(\ell^2(\Xi_L^>))$, $\Xi_L^> := \frac{2\pi\mathbb{Z}^d}{L} \setminus \{\mathbf{0}\}$

$$H_{\text{eff}}^L = H_{\text{bg},v}^L + \sqrt{\kappa}H_{3,v}^L + \kappa H_4^L$$
$$H_{\text{bg},v}^L = \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \omega_{\text{bg}}(\mathbf{p}) b_{\mathbf{p}}^* b_{\mathbf{p}}$$

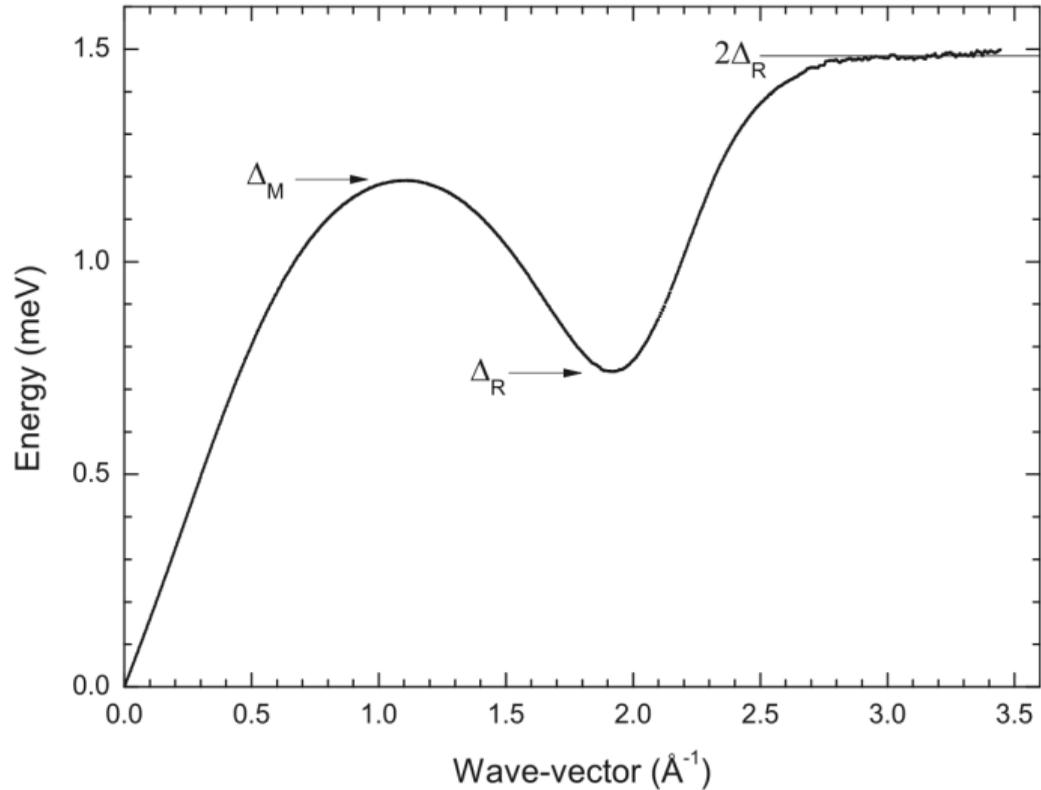
The joint energy-momentum spectrum of $(H_{\text{bg},v}^L, \mathbf{P}^L)$ is given by

$$\sigma(H_{\text{bg},v}^L, \mathbf{P}^L) = \{(\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) \mid \mathbf{k} \in \Xi_L^>\}$$

In the thermodynamic limit the spectrum abs. continuous w.r.t Lebesgue except for $(0, \mathbf{0})$ and the singular spectrum

$$\sigma(H_{\text{bg},v}, \mathbf{P}) \subset [0, +\infty[\times \mathbb{R}^d, \quad \sigma_{\text{sing}}(H_{\text{bg},v}^L, \mathbf{P}^L) = \{(\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) \mid \mathbf{k} \in \mathbb{R}^d\}$$

Quasi-Particle dispersion relation



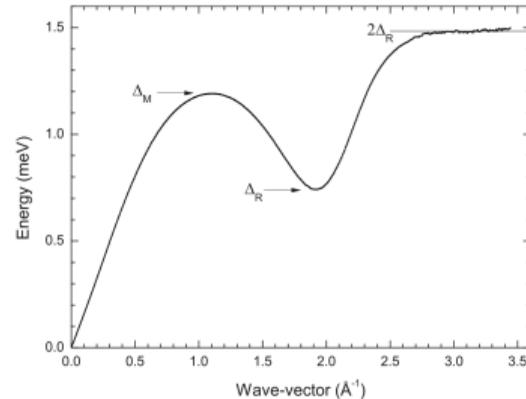
Quasi-Particle dispersion relation



- Dispersion relation:

$$\omega_{\text{bg}}(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|^4}{4} + \frac{v\hat{V}(\mathbf{k})|\mathbf{k}|^2}{\hat{v}(\mathbf{0})}}$$

- Low momenta **Phonons**
 $\omega_{\text{bg}}(\mathbf{k}) \simeq \sqrt{v}|\mathbf{k}|$
- Other excitations: **Maxons, Rotons**
and at the end the **Pitaevskii Plateau**



Godfrin et al. *Dispersion relation of Landau elementary excitations and thermodynamic properties of superfluid ⁴He*. Phys. Rev. B **103**, 104516 (2021)

Decay of quasiparticles



- We expect the interaction terms $H_{3,\nu}^L + H_4^L$ to broaden the spectrum

Decay of quasiparticles



- We expect the interaction terms $H_{3,\nu}^L + H_4^L$ to broaden the spectrum
- Broadening can be described in perturbation theory by computing the **Fermi Golden rule** in the weak coupling limit $\hat{V}(\mathbf{0}) \rightarrow \kappa \hat{V}(\mathbf{0})$, with fixed ν and κ small.
We do this at finite temperature $\frac{1}{\beta} > 0$, $\nu > 0$ and for $L \rightarrow +\infty$

Decay of quasiparticles



- We expect the interaction terms $H_{3,\nu}^L + H_4^L$ to broaden the spectrum
- Broadening can be described in perturbation theory by computing the **Fermi Golden rule** in the weak coupling limit $\hat{V}(\mathbf{0}) \rightarrow \kappa \hat{V}(\mathbf{0})$, with fixed ν and κ small.

We do this at finite temperature $\frac{1}{\beta} > 0$, $\nu > 0$ and for $L \rightarrow +\infty$

- For $\frac{1}{\beta} > 0$, one has to replace the Hamiltonian by the **Liouvillean** in the **Araki-Woods** representations

There will arise naturally two kinds of excitations built over the KMS state: **left** and **right Bogoliubov quasiparticles**

Decay of quasiparticles



- **Two distinct processes** at order κ^1

$$-\gamma_B(\mathbf{k}, \beta, \nu) - \gamma_L(\mathbf{k}, \beta, \nu)$$

Decay of quasiparticles

- Two distinct processes at order κ^1

$$-\gamma_B(\mathbf{k}, \beta, \nu) - \gamma_L(\mathbf{k}, \beta, \nu)$$

- Beliaev damping γ_B** : describes decay of one left particles into two $L \rightarrow L + L$ - Relevant for $\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0$

$$\gamma_B(\mathbf{k}, \beta, \nu) = \frac{3\widehat{V}(\mathbf{0})\nu^{\frac{3}{2}}}{640\pi} \frac{|\mathbf{k}|^5}{\nu^{\frac{5}{2}}} (1 + O(|\mathbf{k}|^2\nu^{-1} + (\beta\sqrt{\nu}|\mathbf{k}|)^{-3}))$$

Decay of quasiparticles

- Two distinct processes at order κ^1

$$-\gamma_B(\mathbf{k}, \beta, \nu) - \gamma_L(\mathbf{k}, \beta, \nu)$$

- Beliaev damping γ_B** : describes decay of one left particles into two $L \rightarrow L + L$ - Relevant for $\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0$

$$\gamma_B(\mathbf{k}, \beta, \nu) = \frac{3\widehat{V}(\mathbf{0})\nu^{\frac{3}{2}}}{640\pi} \frac{|\mathbf{k}|^5}{\nu^{\frac{5}{2}}} (1 + O(|\mathbf{k}|^2\nu^{-1} + (\beta\sqrt{\nu}|\mathbf{k}|)^{-3}))$$

- Landau damping γ_L** : describes decay of left particles into one right and one left $L \rightarrow L + R$ - Relevant for $\beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0$

$$\gamma_L(\mathbf{k}, \beta, \nu) = \frac{3\pi^3\widehat{V}(\mathbf{0})\nu^{\frac{3}{2}}}{40} \frac{|\mathbf{k}|}{\sqrt{\nu}} \frac{1}{(\beta\nu)^4} (1 + O(\beta\sqrt{\nu}|\mathbf{k}| + (\beta\nu)^{-2}))$$

Brief review of the literature



- 1947 Bogoliubov, N., N. J., Phys. (USSR) → First description of low energy **interacting Bose gas**
- 1958 Beliaev, S., T. Sov. Phys. JETP → **Low energy QFT** and computation of **Beliaev Damping** $\frac{1}{\beta} = 0$
- 1960 Mohling, F., Sirlin, A. Phys. Rev → **Perturbative computations** with **hard sphere gas**
- 1965 Hohenberg, P., C., Martin, P., C. Ann. Phys. → First computation of **Landau Damping** $\frac{1}{\beta} > 0$
- 1997 Pitaevskii, L., P., Stringari, S. Phys. Lett. A → First application of the **Fermi golden rule**
- 2024 Dereziński, J., Li, B., Napiorkowski. M, J. Stat. Phys. → Revival of the topic in a rigorous fashion

Derivation of the model

In momentum representation \rightarrow we separate the **zero mode** $\Xi_L = \Xi_L^> \oplus \{\mathbf{0}\}$,
 $\Gamma_s(\Xi_L) \simeq \Gamma_s(\mathbb{C}) \otimes \Gamma_s(\ell^2(\Xi_L^>))$

$$\begin{aligned}
 H^L &= \frac{\hat{V}(\mathbf{0})}{2L^d} N_0(N_0 - 1) + \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{\mathbf{p}^2}{2} + \frac{N_0}{L^d} (\hat{V}(\mathbf{p}) + \hat{V}(\mathbf{0})) \right) a_{\mathbf{p}}^* a_{\mathbf{p}} \\
 &+ \frac{1}{2L^d} \sum_{\mathbf{p} \in \Xi_L^>} \left(\hat{V}(\mathbf{p}) a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_{\mathbf{p}} a_{-\mathbf{p}} + \text{h. c.} \right) \\
 &+ \frac{1}{L^d} \sum_{\mathbf{q}, \mathbf{p}, \mathbf{q}+\mathbf{p} \in \Xi_L^>} \left(\hat{V}(\mathbf{q}) a_{\mathbf{0}}^* a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{q}} a_{\mathbf{p}} + \text{h. c.} \right) \\
 &+ \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k} \in \Xi_L^>}} \hat{V}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}.
 \end{aligned}$$

Extended space formalism



- [Dereziński, Napiórkowski '14, Lewin, Nam, Serfaty, Solovej '15]

Embed $\Gamma_s(\mathbb{C}) \rightarrow \ell^2(\mathbb{Z})$ and **Define** the unitary

$$U|N_0\rangle \otimes \Psi^> = |N_0 + 1\rangle \otimes \Psi^>, \quad U^*|N_0\rangle \otimes \Psi^> = |N_0 - 1\rangle \otimes \Psi^>$$

The rotated creation/annihilation and extended number operators

$$\tilde{a}_{\mathbf{k}} = U a_{\mathbf{k}}, \quad \tilde{a}_{\mathbf{k}}^* = U^* a_{\mathbf{k}}^*$$

$$U^* a_0 = \sqrt{N_0^+}, \quad N_0^{\text{ext}}|N_0\rangle \otimes \Psi^> = N_0|N_0\rangle \otimes \Psi^>$$

Extended space formalism

- o [Dereziński, Napiórkowski '14, Lewin, Nam, Serfaty, Solovej '15]

Embed $\Gamma_s(\mathbb{C}) \rightarrow \ell^2(\mathbb{Z})$ and **Define** the unitary

$$U|N_0\rangle \otimes \Psi^> = |N_0 + 1\rangle \otimes \Psi^>, \quad U^*|N_0\rangle \otimes \Psi^> = |N_0 - 1\rangle \otimes \Psi^>$$

The **rotated** creation/annihilation and extended number operators

$$\tilde{a}_{\mathbf{k}} = U a_{\mathbf{k}}, \quad \tilde{a}_{\mathbf{k}}^* = U^* a_{\mathbf{k}}^*$$

$$U^* a_0 = \sqrt{N_0^+}, \quad N_0^{\text{ext}}|N_0\rangle \otimes \Psi^> = N_0|N_0\rangle \otimes \Psi^>$$

- o The **extended Hamiltonian** becomes

$$H^{L,\text{ext}} = H_{\text{bg},\nu}^L + H_{3,\nu}^L + H_4^L + R_3^L + R_4^L$$

$H_{\text{bg},\nu}^L, H_{3,\nu}^L, H_4^L$ obtained from H_L by replacing $a_0 \rightarrow \sqrt{N - N^>}$

c-number condensate Hamiltonian



- The quadratic Hamiltonian is given by

$$H_{\text{bg},v}^L = \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{|\mathbf{p}|^2}{2} + \frac{v\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) N_{\mathbf{p}} + \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \frac{v\hat{V}(\mathbf{p})}{2\hat{V}(\mathbf{0})} (a_{\mathbf{p}}^* a_{-\mathbf{p}}^* + \text{h.c.})$$

c-number condensate Hamiltonian

- The quadratic Hamiltonian is given by

$$H_{\text{bg},v}^L = \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{|\mathbf{p}|^2}{2} + \frac{v\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) N_{\mathbf{p}} + \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \frac{v\hat{V}(\mathbf{p})}{2\hat{V}(\mathbf{0})} (a_{\mathbf{p}}^* a_{-\mathbf{p}}^* + \text{h.c.})$$

- Diagonalized by taking $a_{\mathbf{k}} = c_{\mathbf{k}} b_{\mathbf{k}} - s_{\mathbf{k}} b_{-\mathbf{k}}^*$

$$c_{\mathbf{k}} = \frac{\sqrt{\frac{|\mathbf{k}|^2}{2} + v\hat{V}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{k})}}{\sqrt{2\omega_{\text{bg}}(\mathbf{k})}}, \quad s_{\mathbf{k}} = \sqrt{c_{\mathbf{k}}^2 - 1}$$

c-number condensate Hamiltonian

- The **quadratic Hamiltonian** is given by

$$H_{\text{bg},v}^L = \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{|\mathbf{p}|^2}{2} + \frac{v\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) N_{\mathbf{p}} + \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \frac{v\hat{V}(\mathbf{p})}{2\hat{V}(\mathbf{0})} (a_{\mathbf{p}}^* a_{-\mathbf{p}}^* + \text{h.c.})$$

- **Diagonalized** by taking $a_{\mathbf{k}} = c_{\mathbf{k}} b_{\mathbf{k}} - s_{\mathbf{k}} b_{-\mathbf{k}}^*$

$$c_{\mathbf{k}} = \frac{\sqrt{\frac{|\mathbf{k}|^2}{2} + v\hat{V}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{k})}}{\sqrt{2\omega_{\text{bg}}(\mathbf{k})}}, \quad s_{\mathbf{k}} = \sqrt{c_{\mathbf{k}}^2 - 1}$$

- **Ground state** constructed as

$$\Omega_{\text{bg}} = \prod_{\mathbf{p} \in \Xi_L^>} c_{\mathbf{k}}^{-\frac{1}{4}} \exp \left\{ -\frac{1}{2} \frac{s_{\mathbf{k}}}{c_{\mathbf{k}}} a_{\mathbf{k}}^* a_{-\mathbf{k}}^* \right\} |N\rangle \otimes \Omega^>$$

Remainder terms of the Hamiltonian



The operators R_3^L, R_4^L contain terms like

$$N_0^+ - N_0^{\text{ext}} = |N_0^{\text{ext}}| P(N_0^{\text{ext}} \leq 0)$$

[Dereziński, P. upcoming] It is possible to show that

$$\lim_{L \rightarrow +\infty} (b_{\mathbf{p}_1}^* \dots b_{\mathbf{p}_n}^* \Omega_{\text{bg}} | R_i^L b_{\mathbf{p}_1}^* \dots b_{\mathbf{p}_n}^* \Omega_{\text{bg}}) = 0$$

If we only **assume**

$$n_0 = \lim_{L \rightarrow +\infty} \frac{1}{L^d} (\Omega_{\text{bg}} | N_0^{\text{ext}} \Omega_{\text{bg}}) = n - \int \frac{d\mathbf{k}}{(2\pi)^d} s_{\mathbf{k}}^2 > 0$$

Thus, **we remain with** $H^L = H_{\text{bg},v}^L + H_{3,v}^L + H_4^L$

Finite temperature formalism



Algebraic formalism : we study the W^* -algebra $B(\Gamma_s(\ell^2(\Xi_L^>)))$

At positive temperature $\frac{1}{\beta} > 0$ we expect to be reasonable to study perturbations of the unperturbed KMS state

$$\Omega_\beta = \frac{e^{-\frac{\beta H_{\text{bg},v}^L}{2}}}{\text{Tr}\left\{e^{-\beta H_{\text{bg},v}^L}\right\}^{\frac{1}{2}}} \in B^2(\Gamma_s(\ell^2(\Xi_L^>)))$$

This can be done by moving to the **standard representation** of $B(\Gamma_s(\ell^2(\Xi_L^>)))$ on the Hilbert Schmidt operators $B^2(\Gamma_s(\ell^2(\Xi_L^>)))$

$$(\Omega|\Psi) := \text{Tr}\Omega^*\Psi, \quad \Omega, \Psi \in B^2(\Gamma_s(\ell^2(\Xi_L^>)))$$

In particular, Ω_β is now a **(squeezed) vector**

Standard representation



Two commuting representations, the **left**

$$\pi_l(A)\Psi := A\Psi, \quad A \in B(\Gamma_s(\ell^2(\Xi_L^>))), \Psi \in B^2(\Gamma_s(\ell^2(\Xi_L^>)))$$

and the (antilinear) **right**

$$\pi_r(A)\Psi := \Psi A^*, \quad A \in B(\Gamma_s(\ell^2(\Xi_L^>))), \Psi \in B^2(\Gamma_s(\ell^2(\Xi_L^>)))$$

They are interchanged by the Anti-linear involution

$$J\pi_l(A)J^* = \pi_r(A^*)$$

The **free dynamics** is generated by the **Liouvillean** $L_{\text{bg},\nu} = \left[H_{\text{bg},\nu}^L, \cdot \right]$

Ω_β is the **standard representative** of the KMS state

Araki–Woods representations [Araki, Woods '63, Dereziński, Gerard '13]

$$B^2(\Gamma_s(\ell^2(\mathfrak{E}_L^>))) \simeq \Gamma_s(\ell^2(\mathfrak{E}_L^>)) \otimes \overline{\Gamma_s(\ell^2(\mathfrak{E}_L^>))} \simeq \Gamma_s(\ell^2(\mathfrak{E}_L^>)) \oplus \overline{\ell^2(\mathfrak{E}_L^>)}$$

The last identification **mixes** left and right components:

For $f \in \ell^2(\mathfrak{E}_L^>)$, $\rho = e^{-\beta\omega_{\text{bg}}(\mathbf{P})}(1 - e^{-\beta\omega_{\text{bg}}(\mathbf{P})})^{-1}$

$$b_{\beta,l}^*(f) = b_l^*((1 + \rho)^{\frac{1}{2}}f) + b_r(\rho^{\frac{1}{2}}\bar{f})$$

$$b_{\beta,r}^*(f) = b_r^*((1 + \rho)^{\frac{1}{2}}f) + b_l(\rho^{\frac{1}{2}}\bar{f})$$

The **KMS state** mapped to the **vacuum** state $\Omega \rightarrow b_l(f)\Omega = b_r(f)\Omega = 0$.

For the standard basis of $\ell^2(\Gamma_s(\Xi_L^>))$

$$b_{\beta,l}^*(\mathbf{k}) = (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_l^*(\mathbf{k}) + (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_r(\mathbf{k}),$$

$$b_{\beta,l}(\mathbf{k}) = (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_l(\mathbf{k}) + (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_r^*(\mathbf{k}),$$

$$b_{\beta,r}^*(\mathbf{k}) = (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_l(\mathbf{k}) + (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_r^*(\mathbf{k}),$$

$$b_{\beta,r}(\mathbf{k}) = (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_l^*(\mathbf{k}) + (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_r(\mathbf{k})$$

We can construct finite number of excitations states $b_l^*(\mathbf{k}_1) \dots b_r^*(\mathbf{p}_m)\Omega$

Perturbation theory over a thermal background



- We do perturbation theory [Derezisński, Jaksic, Pillet '03] with unperturbed vectors

$$b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega,$$

Perturbation theory over a thermal background



- We do perturbation theory [Derezisński, Jaksic, Pillet '03] with unperturbed vectors

$$b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega,$$

- The Hamiltonian $H_{\text{bg},\nu}$ is substituted by the Liouvillean $L_{\text{bg},\nu}$

$$L_{\text{bg},\nu} = \sum_{\mathbf{p} \in \Xi_L^>} \omega_{\text{bg}}(\mathbf{p})(b_l^*(\mathbf{p})b_l(\mathbf{p}) - b_r^*(\mathbf{k})b_r^*(\mathbf{k}))$$

Similarly $H_{3,\nu}^L \rightarrow L_{3,\nu}$ and $H_4^L \rightarrow L_4$ are obtained by including **right particles**

Perturbation theory over a thermal background



- We do perturbation theory [Derezisński, Jaksic, Pillet '03] with unperturbed vectors

$$b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega,$$

- The Hamiltonian $H_{\text{bg},\nu}$ is substituted by the Liouvillean $L_{\text{bg},\nu}$

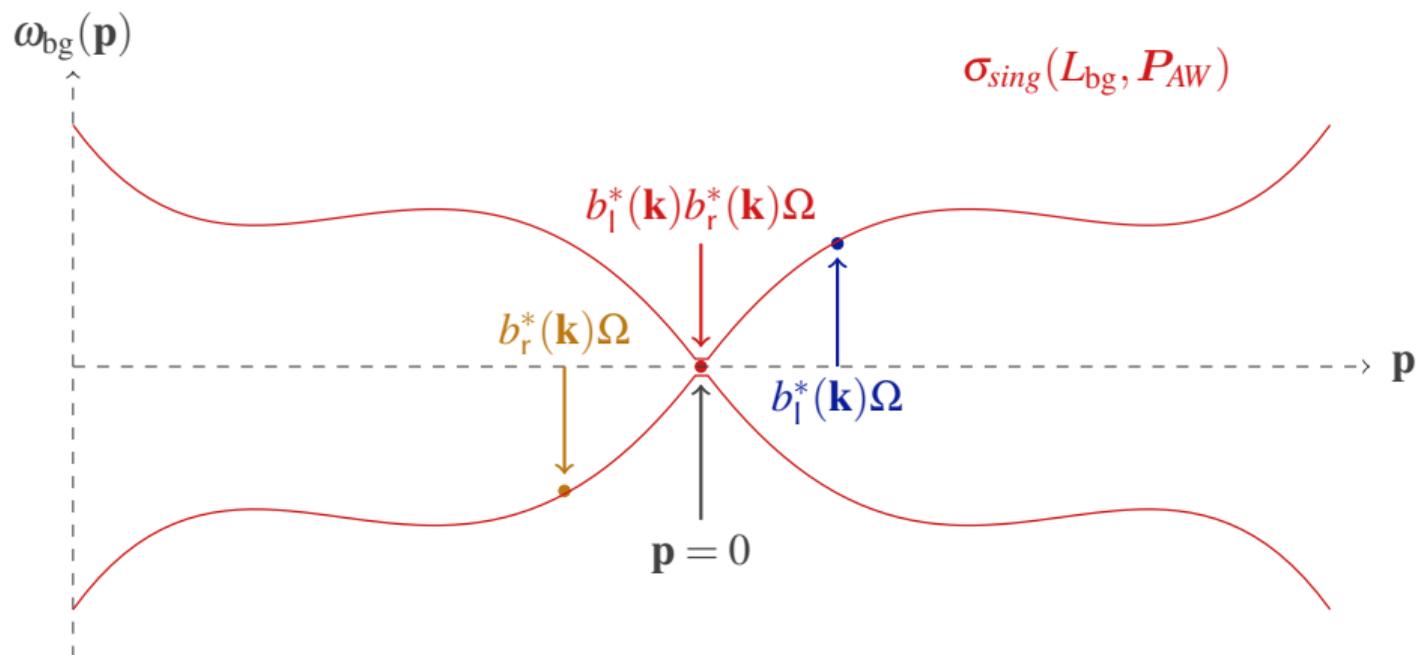
$$L_{\text{bg},\nu} = \sum_{\mathbf{p} \in \Xi_L^>} \omega_{\text{bg}}(\mathbf{p})(b_l^*(\mathbf{p})b_l(\mathbf{p}) - b_r^*(\mathbf{k})b_r^*(\mathbf{k}))$$

Similarly $H_{3,\nu}^L \rightarrow L_{3,\nu}$ and $H_4^L \rightarrow L_4$ are obtained by including **right particles**

- In thermodynamic limit $\sigma(L_{\text{bg},\nu}, P_{AW}) = \mathbb{R}^{1+d}$ while

$$\sigma_{\text{sing}}(L_{\text{bg},\nu}, P_{AW}) = \{(\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) | \mathbf{k} \in \mathbb{R}^d\} \cup \{(-\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) | \mathbf{k} \in \mathbb{R}^d\}$$

Unperturbed thermal vectors

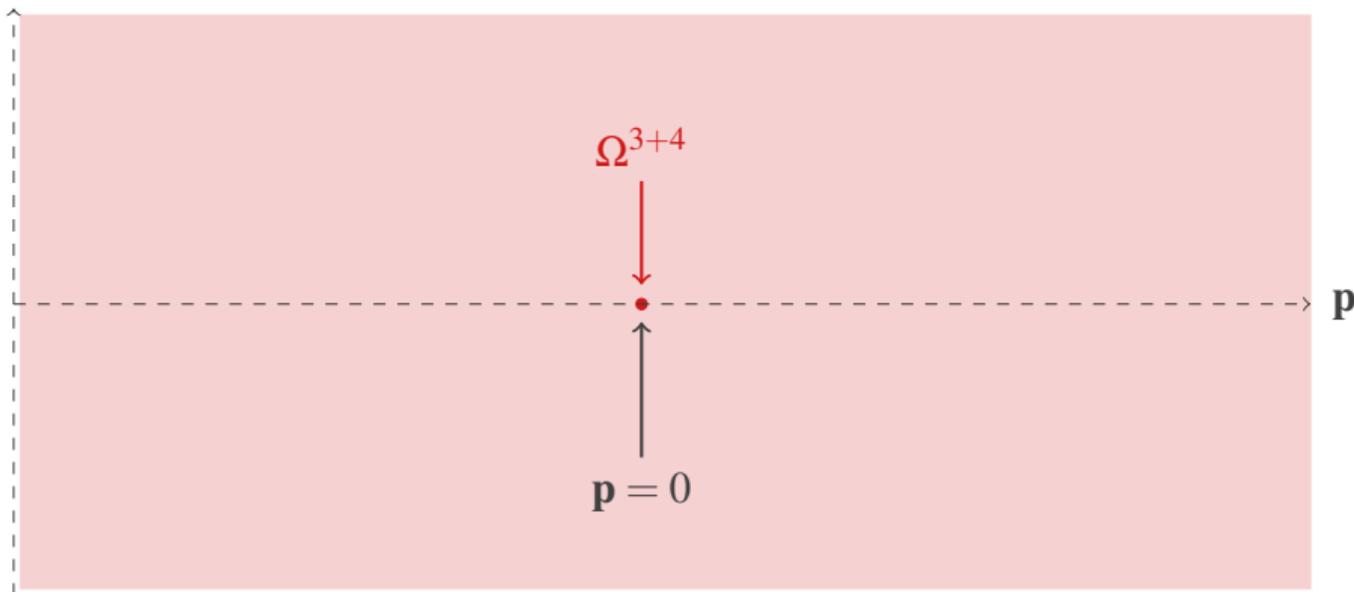


Perturbed thermal vectors



In **thermodynamic limit** one expect perturbed states to vanish

$$\sigma(L_{\text{bg},v} + L_{3,v} + L_4)$$



Finite temperature computations

- Inspired by ideas of **Jaksic–Pillet ('96)**, we compute second-order perturbation theory

$$\xi(\mathbf{k}) := \lim_{\varepsilon \rightarrow 0^+} \lim_{L \rightarrow \infty} \left\{ - \left(L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \mid (L_{\text{bg}} - i\varepsilon)^{-1} L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \right) \right. \\ \left. - (L_{3,v} \Omega \mid (L_{\text{bg},v} - i\varepsilon)^{-1} L_{3,v} \Omega) \right\}$$

Finite temperature computations

- Inspired by ideas of **Jaksic–Pillet ('96)**, we compute second-order perturbation theory

$$\xi(\mathbf{k}) := \lim_{\varepsilon \rightarrow 0^+} \lim_{L \rightarrow \infty} \left\{ - \left(L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \mid (L_{bg} - i\varepsilon)^{-1} L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \right) \right. \\ \left. - (L_{3,v} \Omega \mid (L_{bg,v} - i\varepsilon)^{-1} L_{3,v} \Omega) \right\}$$

- The subtraction \uparrow is necessary because

$$\left(L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \mid (L_{bg,v} - i\varepsilon)^{-1} L_{3,v} b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \right) = \text{finite terms} \\ + (\Omega \mid L_{3,v} (L_{bg,v} - i\varepsilon) L_{3,v} \Omega),$$

$$(\Omega \mid L_{3,v} (L_{bg,v} - i\varepsilon) L_{3,v} \Omega) \text{ diverges as } L^d, \lim_{\varepsilon \rightarrow 0^+} (\Omega \mid L_{3,v} (L_{bg,v} - i\varepsilon) L_{3,v} \Omega) = 0$$

Finite temperature computations

- Inspired by ideas of **Jaksić–Pillet ('96)**, we compute second-order perturbation theory

$$\xi(\mathbf{k}) := \lim_{\varepsilon \rightarrow 0^+} \lim_{L \rightarrow \infty} \left\{ - \left(L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \middle| (L_{bg} - i\varepsilon)^{-1} L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \right) \right. \\ \left. - (L_{3,v} \Omega \middle| (L_{bg,v} - i\varepsilon)^{-1} L_{3,v} \Omega) \right\}$$

- The subtraction \uparrow is necessary because

$$\left(L_3 b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \middle| (L_{bg,v} - i\varepsilon)^{-1} L_{3,v} b_l(\mathbf{k})^* b_r^*(\mathbf{k}) \Omega \right) = \text{finite terms} \\ + (\Omega \middle| L_{3,v} (L_{bg,v} - i\varepsilon) L_{3,v} \Omega),$$

$(\Omega \middle| L_{3,v} (L_{bg,v} - i\varepsilon) L_{3,v} \Omega)$ diverges as L^d , $\lim_{\varepsilon \rightarrow 0^+} (\Omega \middle| L_{3,v} (L_{bg,v} - i\varepsilon) L_{3,v} \Omega) = 0$

RMK: We did not include κL_4 because they do not contribute to this order

Assumptions on the potential



- For the **Beliaev damping** we need the following (valid for all $v \in]0, W[$)

$$\frac{v}{\widehat{V}(\mathbf{0})} \frac{d^2 \widehat{V}(\mathbf{0})}{dk^2} > -1; \quad (1a)$$

$$\widehat{V} \text{ is } C^5 \text{ in a neighborhood of } \mathbf{k} = \mathbf{0}; \quad (1b)$$

$$V \text{ is rotationally invariant} \quad (1c)$$

Assumptions on the potential



- For the **Beliaev damping** we need the following (valid for all $v \in]0, W[$)

$$\frac{v}{\widehat{V}(\mathbf{0})} \frac{d^2 \widehat{V}(\mathbf{0})}{dk^2} > -1; \quad (1a)$$

$$\widehat{V} \text{ is } C^5 \text{ in a neighborhood of } \mathbf{k} = \mathbf{0}; \quad (1b)$$

$$V \text{ is rotationally invariant} \quad (1c)$$

- For the **Landau damping** in addition

$$\widehat{V} \text{ is } C^1 \quad (2a)$$

$$\frac{d\omega_{\text{bg}}(k)}{dk} = 0 \text{ has at most a finite number of solutions} \quad (2b)$$

Proposition [Dereziński, P. '26]: in **second order perturbation theory** there are purely imaginary correction

$$\begin{aligned}\gamma_B(\mathbf{k}, \beta, \nu) &= \frac{\pi}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2 \delta(\omega_{bg}(\mathbf{k}) - \omega_{bg}(\mathbf{p}) - \omega_{bg}(\mathbf{k} - \mathbf{p})) \\ &\quad \times \frac{(1 - e^{\frac{\beta}{2}(\omega_{bg}(\mathbf{k}) + \omega_{bg}(\mathbf{p}) + \omega_{bg}(\mathbf{k} - \mathbf{p}))})^2}{(e^{\beta\omega_{bg}(\mathbf{k})} - 1)(e^{\beta\omega_{bg}(\mathbf{p})} - 1)(e^{\beta\omega_{bg}(\mathbf{k} - \mathbf{p})} - 1)}, \\ \gamma_L(\mathbf{k}, \beta, \nu) &= \pi \int \frac{d\mathbf{p}}{(2\pi)^3} j(\mathbf{p} - \mathbf{k}; \mathbf{k}, \mathbf{p})^2 \delta(\omega_{bg}(\mathbf{k} - \mathbf{p}) - \omega_{bg}(\mathbf{p}) - \omega_{bg}(\mathbf{k})) \\ &\quad \times \frac{(e^{\frac{\beta}{2}(\omega_{bg}(\mathbf{k}) + \omega_{bg}(\mathbf{k} - \mathbf{p}))} - e^{\frac{\beta}{2}\omega_{bg}(\mathbf{p})})^2}{(e^{\beta\omega_{bg}(\mathbf{k})} - 1)(e^{\beta\omega_{bg}(\mathbf{p})} - 1)(e^{\beta\omega_{bg}(\mathbf{k} - \mathbf{p})} - 1)}\end{aligned}$$

Dirac's delta prescriptions



We work in the **small momentum** limit $\frac{|\mathbf{k}|}{\sqrt{v}} \rightarrow 0$

- The **Beliaev damping delta** $\delta(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k} - \mathbf{p}))$ bounds the momentum $0 \leq |\mathbf{p}| \leq |\mathbf{k}| \rightarrow$ we only have to integrate over a compact region (close to $\mathbf{p} = 0$)

Dirac's delta prescriptions



We work in the **small momentum** limit $\frac{|\mathbf{k}|}{\sqrt{v}} \rightarrow 0$

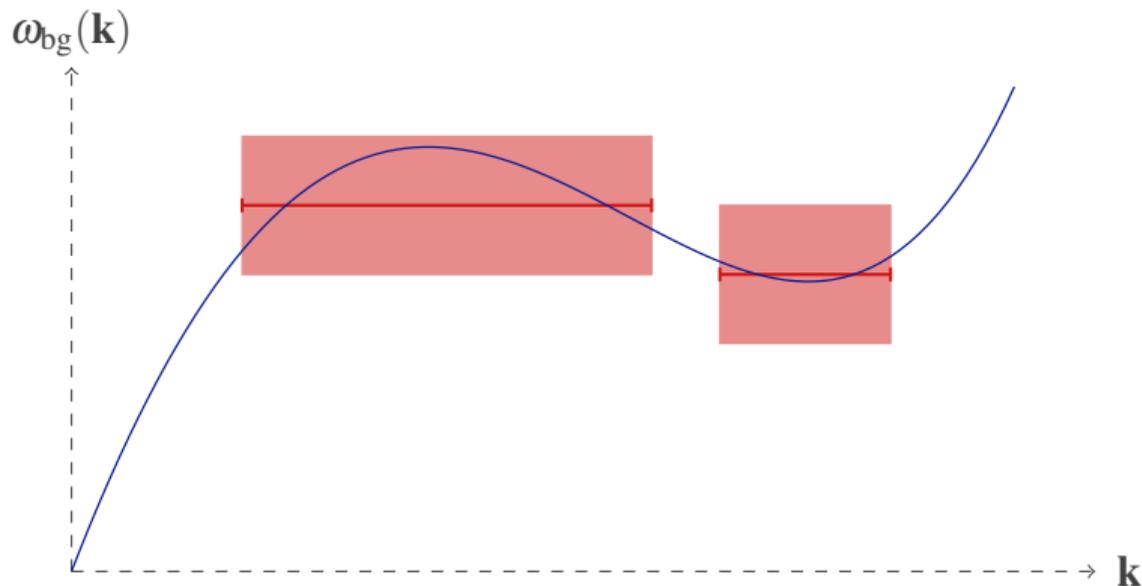
- The **Beliaev damping delta** $\delta(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k} - \mathbf{p}))$ bounds the momentum $0 \leq |\mathbf{p}| \leq |\mathbf{k}| \rightarrow$ we only have to integrate over a compact region (close to $\mathbf{p} = 0$)
- The **Landau damping delta** $\delta(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k} - \mathbf{p}))$ can allow for arbitrarily high momenta...

$\omega_{\text{bg}}(\mathbf{p})$ has maxima and minima in the high momenta region \rightarrow this could cause problems for the well-definiteness of the delta

Dominant contributions to our integrals



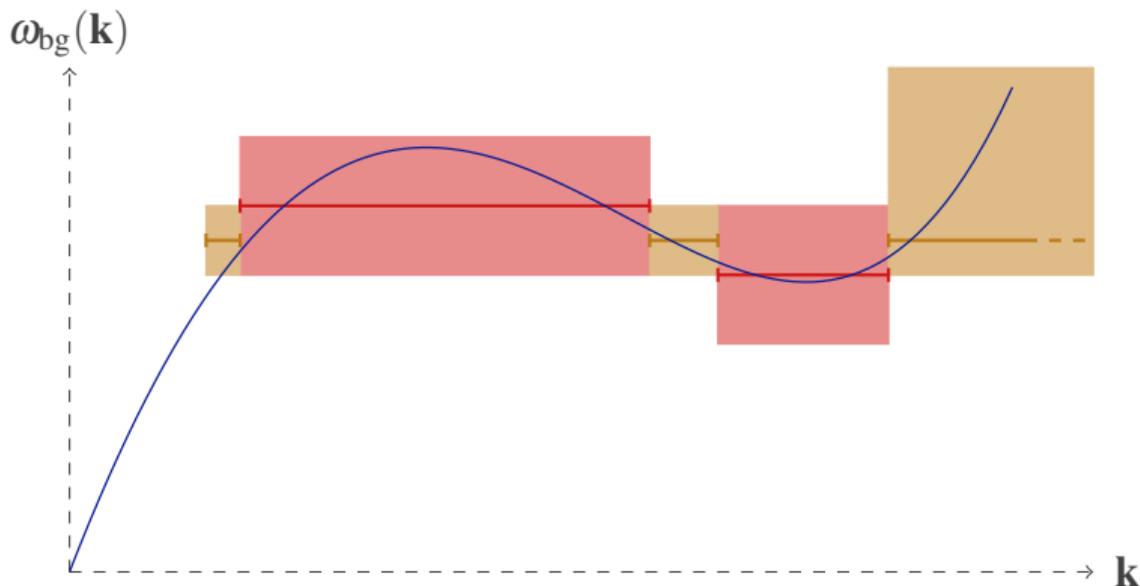
Proposition [Dereziński, P. '26]: the red intervals are outside the support of $\delta(\dots)$



Dominant contributions to our integrals



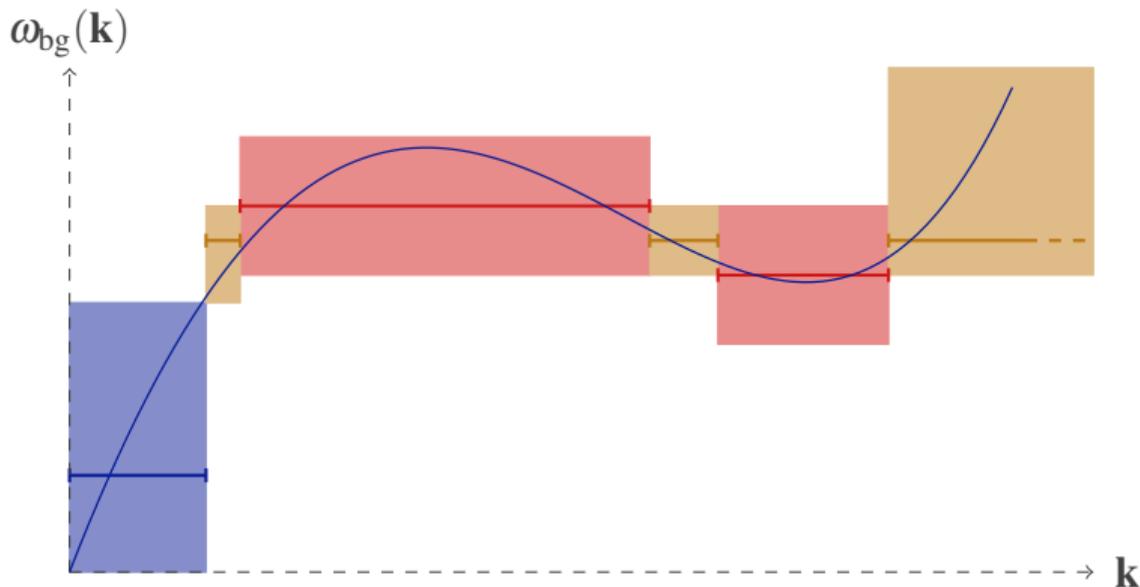
Proposition [Dereziński, P. '26]: the yellow regions are exponentially damped as $e^{-c\beta\nu}$, $\beta\nu \rightarrow +\infty$



Dominant contributions to our integrals



Theorem [Dereziński, P. '26]: the phonon region is the only one that matters



Computations of the imaginary component recover the **predicted expression** in the correct limits after substituting $\hat{V}(\mathbf{0}) \leftrightarrow 4\pi\alpha$

$$\circ \gamma_B(\mathbf{k}, \beta, \nu) = \frac{3\hat{V}(\mathbf{0})\nu^{3/2}}{640\pi} \frac{|\mathbf{k}|^5}{\nu^{5/2}} \left(1 + O\left(\frac{1}{(\beta\sqrt{\nu}|\mathbf{k}|)^3}\right) + O\left(\frac{|\mathbf{k}|^2}{\nu}\right) \right)$$

$$\text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0$$

$$\circ \gamma_L(\mathbf{k}; \beta, \nu) = \frac{3\pi^3\hat{V}(\mathbf{0})\nu^{3/2}}{40} \frac{|\mathbf{k}|}{\sqrt{\nu}} \frac{1}{(\beta\nu)^4} \left(1 + O(\beta\sqrt{\nu}|\mathbf{k}|) + O\left(\frac{1}{(\beta\nu)^2}\right) \right)$$

$$\text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu}, \beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0$$

These results are universal in the choice of \hat{V} , in the sense that within our hypothesis they only depend on $\hat{V}(\mathbf{0})$



Thank you for your attention!



Thank you for your attention?

Green functions point of view I



- Formalism of **Green functions** : A_1, A_2 operators, $t \in \mathbb{R}$ time and $\frac{1}{\beta} > 0$ temperature

Two-point time-ordered function :

$$G_{\beta}(A_2, A_1; t) = i(\Omega | T(A_2(t)A_1(0)) \Omega)$$

or in **energy representation**

$$\begin{aligned} G_{\beta}(A_2, A_1; E) &= \lim_{\varepsilon \rightarrow 0^+} i \int_0^{+\infty} (\Omega | A_2 e^{-itL_{\text{bg},v} - \varepsilon t - iEt} A_1 \Omega) \\ &\quad + \lim_{\varepsilon \rightarrow 0^+} \int_0^{+\infty} (\Omega | A_1 e^{-itL_{\text{bg},v} - \varepsilon t + iEt} A_2 \Omega) \\ &= (\Omega | A_2 (L_{\text{bg},v} - i0 + E)^{-1} A_1 \Omega) + (\Omega | A_1 (L_{\text{bg},v} - i0 - E)^{-1} A_2 \Omega) \end{aligned}$$

Green functions point of view II



As an example $A_1 = b_{\beta,l}(\mathbf{k})$ and $A_2 = b_{\beta,l}(\mathbf{k})^*$ give

$$G_{\beta}(b_{\beta,l}(\mathbf{k})^*, b_{\beta,l}(\mathbf{k}); E) = \frac{1}{\omega_{\text{bg}}(\mathbf{k}) - i0 - E}$$

For interacting models

$$G_{\beta}^{3+4}(b_{\beta,l}(\mathbf{k})^*, b_{\beta,l}(\mathbf{k}); E) = Z(E, \mathbf{k}) \frac{1}{\omega(\mathbf{k}) - E} + \text{regular}, \quad \text{Im}(\omega(\mathbf{k})) > 0$$

The shift can be computed in perturbation theory for $b_1^*(\mathbf{k})\Omega$

Defining

$$\delta(\mathbf{k}) := \lim_{\varepsilon \rightarrow 0^+} \lim_{L \rightarrow \infty} \left\{ \left\{ \begin{aligned} &(b_1^*(\mathbf{k})\Omega | L_4 b_1^*(\mathbf{k})\Omega) \\ &- \left(L_{3,v} b_1^*(\mathbf{k})\Omega | (L_{\text{bg},v} - \omega_{\text{bg}}(\mathbf{k}) - i\varepsilon)^{-1} L_{3,v} b_1^*(\mathbf{k})\Omega \right) \\ &- (\Omega | L_4 \Omega) + (L_{3,v}\Omega | (L_{\text{bg},v} - i\varepsilon)^{-1} L_{3,v}\Omega) \end{aligned} \right\} \right\}$$

Consistent results for the imaginary part

$$\text{Im}(\delta(\mathbf{k})) = 2 \text{Im}(\xi(\mathbf{k}))$$

[**Dereziński, P. upcoming**] \rightarrow we are completing the study of $\text{Re}(\delta(\mathbf{k}))$ (Spoiler: results consistent with physics literature at $\frac{1}{\beta} = 0$)



Thank you (again) for your attention!