Phonon Dispersion beyond the Bogoliubov Approximation

Joint work with Jan Dereziński

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Introduction to the physics

Consider an N-particles, zero temperature, homogeneous Bose gas in a large periodic box $]-\frac{L}{2},\frac{L}{2}]^d$ $d \geq 2$ with mean density n and two-body potential $V(\mathbf{x})$

- For weakly interacting gasses, there is an effective description in terms of quasiparticle excitations with dispersion relation $\mathbf{k} \to \omega_{\mathrm{bg}}(\mathbf{k})$. This description is obtained by diagonalizing a quadratic Hamiltonian that naturally arises in perturbation theory.

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- The perturbative computation is organized in powers of $\sqrt{\kappa}$ after rescaling $V(\mathbf{x}) \to \kappa V(\mathbf{x})$ with κ small, while keeping $\nu := \kappa n \widehat{V}(\mathbf{0})$ fixed

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- The perturbative computation is organized in powers of $\sqrt{\kappa}$ after rescaling $V(\mathbf{x}) \to \kappa V(\mathbf{x})$ with κ small, while keeping $\nu := \kappa n \widehat{V}(\mathbf{0})$ fixed
- These quasiparticles are called phonons. Their dispersion relation is linear for small momenta: $\omega_{\mathrm{bg}}(\mathbf{k}) \simeq \sqrt{\nu}|\mathbf{k}|$. It is believed that interaction terms will shift

$$\omega_{\mathrm{bg}}(\mathbf{k}) \to \omega_{\mathrm{bg}}(\mathbf{k}) + \delta(\mathbf{k})$$

while preserving the linear behavior at small momenta Hugenholtz & Pines '59

Objective of the talk

In this talk, we will compute the second-order Rayleigh-Schrödinger perturbative correction $(\sqrt{\kappa})^2\delta^{(2)}(\mathbf{k})$ in two different frameworks

1. Canonical: The number of particle is fixed to N, with $\frac{N}{L^d} = n$ as $L \to +\infty$. The number operator satisfies $\mathscr{N} = \mathbb{1}N$
2. grand-canonical: The Hamiltonian is shifted by a chemical potential term

$$H \to H - \mu\mathscr{N}$$

   μ is tuned to keep the density of the particles in the ground state $n := \frac{\langle \mathscr{N} \rangle}{L^d}$ fixed

Surprisingly, the canonical $\delta^{(2)}(\mathbf{k})$ and grand-canonical $\delta_\mu^{(2)}(\mathbf{k})$ coincide up to smaller order in $\sqrt{\kappa}$.

Basic hypothesis on the potential

Suppose that

a) $V \in L^1(\mathbb{R}^d)$ and $V(x) \in \mathbb{R}$;

b) $\widehat{V} \in L^2(\mathbb{R}^d)$;

Define

$$\nu := \kappa \widehat{V}(\mathbf{0}) n$$

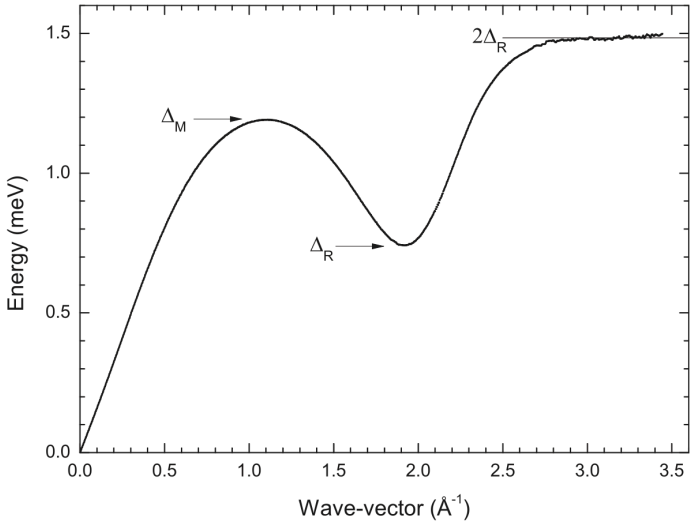
and suppose there exists $\nu_0 \in \mathbb{R}_+$ such that the following hold for all $\nu \in]0, \nu_0]$

c) $\widehat{V}(\mathbf{0}) > 0$ and $\widehat{V}(\mathbf{k}) > -\widehat{V}(\mathbf{0}) \frac{|\mathbf{k}|^2}{2\nu} \rightarrow$ these allow to define $\omega_{\mathrm{bg}}(\mathbf{k})$

d) $V(x) = V(-x)$ which implies $\widehat{V}(\mathbf{k}) = \widehat{V}(-\mathbf{k})$

RMK: The finite-volume potential should be named V_L and we assume $V_L \in L^2$ in $]\frac{L}{2}, \frac{L}{2}]^d$
It is obtained by periodization of V and we **avoid writing** the subscript L

quasiparticle dispersion relation

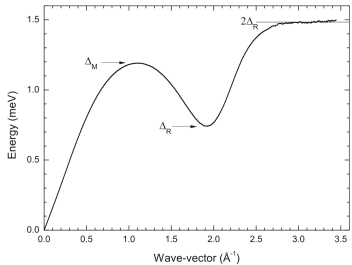


quasiparticle dispersion relation

- Dispersion relation:

$$\omega_{\mathrm{bg}}(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|^4}{4} + \frac{v\widehat{V}(\mathbf{k})|\mathbf{k}|^2}{\widehat{V}(\mathbf{0})}}$$

- Low momenta Phonons $\omega_{\mathrm{bg}}(\mathbf{k}) \simeq \sqrt{v}|\mathbf{k}|$
- Other excitations: Maxons, Rotons and at the end the Pitaevskii Plateau, which is **not described** by our models



Godfrin et al. *Dispersion relation of Landau elementary excitations and thermodynamic properties of superfluid* ^{4}*He.* Phys. Rev. B **103**, 104516 (2021)

Parameters of the theory

- Phonons becomes relevant in the small momenta $\mathbf{k}\sqrt{v}^{-1} \to 0$

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- Phonons becomes relevant in the small momenta $\mathbf{k}\sqrt{\nu}^{-1} \to 0$
- By taking $\nu \to \kappa\nu$ we compute the dilute limit. This corresponds to rescaling the scattering length as

$$\mathfrak{a} = \kappa\mathfrak{a}_0 + \kappa^2\mathfrak{a}_1 + \dots$$

The scattering length substitutes the potential with the formal identification

$$\mathfrak{a}_0 = 4\pi\widehat{V}(\mathbf{0}), \qquad \mathfrak{a}_1 = -\frac{1}{4\pi}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{\widehat{V}(\mathbf{p})^2}{|\mathbf{p}|^2}.$$

In this limit we will show that $\delta^{(2)}(\mathbf{k})$ coincide with the one in Beliaev '57

$\omega_{\mathrm{bg}}(\mathbf{k}) + \kappa\delta^{(2)}(\mathbf{k}) = \sqrt{4\pi n\mathfrak{a}}|\mathbf{k}|\left(1 + \frac{1}{\pi^2}\sqrt{n(4\pi\mathfrak{a})^3}\right)$ The red term is Beliaev's correction

Canonical formalism: the Hamiltonian

We write the finite-volume Hamiltonian

$$H^L = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \sum_{i<j}^{N} V(\mathbf{x}_i - \mathbf{x}_j)$$

in the second quantization formalism

$$\sum_{\mathbf{p}\in\Xi_L}\frac{|\mathbf{p}|^2}{2}a^*_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2L^d}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}\in\Xi_L}\widehat{V}(\mathbf{k})a^*_{\mathbf{p}-\mathbf{k}}a^*_{\mathbf{q}+\mathbf{k}}a_{\mathbf{q}}a_{\mathbf{p}}, \qquad \Xi_L := \frac{2\pi\mathbb{Z}^d}{L}$$

With creation/annihilation operators on Fock space $\Gamma_s(\Xi_L)$:
$a^*_{\mathbf{p}}a_{\mathbf{p}}$, $\mathbf{p}\in\Xi_L^> := \frac{2\pi\mathbb{Z}^d}{L}\setminus\{\mathbf{0}\}$ excited modes and $a^*_{\mathbf{0}}a_{\mathbf{0}}$ zero-modes

Diagonal and off-diagonal interaction

We can write the interaction term as a sum of **Diagonal** and **off-diagonal** modes

$$V_{\text{diag}} = \frac{1}{2L^d} \sum_{\mathbf{p},\mathbf{q}\in\Xi_L} N_{\mathbf{p}} N_{\mathbf{q}} \left(\widehat{V}(\mathbf{0}) + \widehat{V}(\mathbf{q}-\mathbf{p})\right) - \frac{\widehat{V}(\mathbf{0})}{2L^d} \sum_{\mathbf{p}\in\Xi_L} N_{\mathbf{p}}(N_{\mathbf{p}}+1)$$

$$V_{\text{off}} = \frac{1}{2L^d} \sum_{\mathbf{k},\mathbf{q}-\mathbf{p}-\mathbf{k}\in\Xi_L^{>}} \widehat{V}(\mathbf{k}) a^*_{\mathbf{p}+\mathbf{k}} a^*_{\mathbf{q}-\mathbf{k}} a_{\mathbf{q}} a_{\mathbf{p}}.$$

with

$$\frac{1}{L^d} \sum_{\mathbf{p},\mathbf{q},\mathbf{k}\in\Xi_L} \widehat{V}(\mathbf{k}) a^*_{\mathbf{p}-\mathbf{k}} a^*_{\mathbf{q}+\mathbf{k}} a_{\mathbf{q}} a_{\mathbf{p}} = V_{\text{diag}} + V_{\text{off}}$$

We use the shorthands $N_{\mathbf{p}} = a^*_{\mathbf{p}} a_{\mathbf{p}}$ $A_{\mathbf{p}} = A_{-\mathbf{p}} = a_{\mathbf{p}} a_{-\mathbf{p}}$

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- We isolate the **zero-modes** by explicitly writing $a_{\mathbf{0}}$ and $a^*_{\mathbf{0}}$

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- We use the **canonical formalism** to write

$$N_{\mathbf{0}} = N\mathbb{1} - N^{>}, \qquad N^{>} := \sum_{\mathbf{p}\in\Xi_L^{>}} a^*_{\mathbf{p}} a_{\mathbf{p}}$$

Good second-quantized terms

- We isolate the **zero-modes** by explicitly writing $a_{\mathbf{0}}$ and $a_{\mathbf{0}}^*$
- We use the **canonical formalism** to write

$$N_{\mathbf{0}} = N\mathbb{1} - N^{>}, \qquad N^{>} := \sum_{\mathbf{p}\in\Xi_L^{>}} a_{\mathbf{p}}^* a_{\mathbf{p}}$$

Then, the diagonal interaction can be written entirely as an operator acting on $\Gamma_s(\Xi_L^{>})$

$$V_{\text{diag}} = \frac{1}{2L^d}\left(\widehat{V}(\mathbf{0})N(N-1) + 2N\sum_{\mathbf{p}\in\Xi_L^{>}} N_{\mathbf{p}}\widehat{V}(\mathbf{p}) - \sum_{\mathbf{p},\mathbf{q}\in\Xi_L^{>}} N_{\mathbf{q}}\big(2\widehat{V}(\mathbf{p}) - \widehat{V}(\mathbf{p}-\mathbf{q})\big)N_{\mathbf{p}}\right) - \frac{\widehat{V}(\mathbf{0})}{2L^d}\sum_{\mathbf{p}\in\Xi_L^{>}} N_{\mathbf{p}}^2$$

Bad second-quantized terms

The situation is different for the off-diagonal interaction: $a_{\mathbf{0}}$ and $a_{\mathbf{0}}^*$ appear separately

$$
\begin{aligned}
V_{\text{off}} = & \frac{1}{2L^d} \sum_{\mathbf{p} \in \Xi_L^>} \widehat{V}(\mathbf{p}) (A_{\mathbf{0}}^* A_{\mathbf{p}} + \text{h.c.}) \\
& + \frac{1}{L^d} \sum_{\mathbf{p},\mathbf{q},\mathbf{p}+\mathbf{q} \in \Xi_L^>} \widehat{V}(\mathbf{p}) (a_{\mathbf{0}}^* a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{p}} a_{\mathbf{q}} + \text{h.c.}) \\
& + \frac{1}{2L^d} \sum_{\mathbf{p},\mathbf{q},\mathbf{p} \pm \mathbf{q} \in \Xi_L^>} \widehat{V}(\mathbf{p}-\mathbf{q}) A_{\mathbf{p}}^* A_{\mathbf{q}} \\
& + \frac{1}{2L^d} \sum_{\substack{\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{p}+\mathbf{q}, \\ \mathbf{p}+\mathbf{k},\mathbf{q}-\mathbf{k},\mathbf{p}+\mathbf{k}-\mathbf{q} \in \Xi_L^>}} \widehat{V}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}
\end{aligned}
$$

Extended space formalism

Dereziński, Napiórkowski 2014, Lewin, Nam, Serfaty, Solovej 2015
Embed $\Gamma_s(\mathbb{C}) \to \ell^2(\mathbb{Z})$ and define the unitary shift

$$U|N_0) \otimes \Psi^> = |N_0+1) \otimes \Psi^>, \qquad U^*|N_0) \otimes \Psi^> = |N_0-1) \otimes \Psi^>$$

and the rotated creation/annihilation and extended number operators

$$\tilde{a}_{\mathbf{k}} = U a_{\mathbf{k}}, \qquad \tilde{a}^*_{\mathbf{k}} = U^* a^*_{\mathbf{k}}$$
$$U^* a_0 = \sqrt{N_0^+}, \qquad N_0^{\mathrm{ext}}|N_0) \otimes \Psi^> = N_0|N_0) \otimes \Psi^>$$

The extended Hamiltonian $H^{L,\mathrm{ext}}$ is obtained by replacing

$$a_{\mathbf{0}} \to \sqrt{N_{\mathbf{0}}^+}, \qquad N\mathbb{1} = N_{\mathbf{0}}^{\mathrm{ext}} + N^>$$

Canonical Hamiltonian

We obtain the complete finite-volume Hamiltonian

$$\begin{aligned} H^{L,\text{ext}} =& H^L_{\text{bg}} + \sqrt{\kappa} H^L_3 + \kappa H^L_2 + \kappa H^L_{4,2} + \kappa H^L_{4,1} + \kappa H^L_{4,3} + \kappa\sqrt{\kappa} H^L_{3,x} \\ &+ R^L_{4,1} + R^L_{4,2} \end{aligned}$$

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- The free Bogoliubov Hamiltonian

$$H^L_{\mathrm{bg}} := \frac{1}{L^d} \sum_{\mathbf{p}\in\Xi^>_L} \left(\frac{|\mathbf{p}|^2}{2} + \frac{v\widehat{V}(\mathbf{p})}{\widehat{V}(\mathbf{0})} \right) N_{\mathbf{p}} + \frac{1}{L^d} \sum_{\mathbf{p}\in\Xi^>_L} \frac{v\widehat{V}(\mathbf{p})}{2\widehat{V}(\mathbf{0})} \left(\tilde{A}^*_{\mathbf{p}} + \mathrm{h.c.} \right)$$

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- $H^L_3, H^L_2, H^L_{4,i}$ are interaction terms with 2, 3 and 4 excitation operators $a_{\mathbf{p}}$, $a^*_{\mathbf{p}}$

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- $H^L_3, H^L_2, H^L_{4,i}$ are interaction terms with 2, 3 and 4 excitation operators $a_{\mathbf{p}}$, $a^*_{\mathbf{p}}$
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- $H^L_3, H^L_2, H^L_{4,i}$ are interaction terms with 2, 3 and 4 excitation operators $a_{\mathbf{p}}$, $a^*_{\mathbf{p}}$
- $H^L_{3,x}$ contains a more **involved dependence** on $a_{\mathbf{p}}$, $a^*_{\mathbf{p}}$
- $R^L_{4,1}$ and $R^L_{4,2}$ are tail terms supported on the sector $N^{\mathrm{ext}}_{\mathbf{0}} \leq 0$

Bogoliubov ground state and quasiparticles

H^L_{bg} is diagonalized Bogoliubov vacuum Ω_{bg} and quasiparticles $b^*_{\mathbf{k}}\Omega_{\mathrm{bg}}$, $\mathbf{k} \in \Xi^>_L$

$$H^L_{\mathrm{bg}} = \sum_{\mathbf{p}\in\Xi^>_L} \omega_{\mathrm{bg}}(\mathbf{p}) b^*_{\mathbf{p}} b_{\mathbf{p}}, \qquad b_{\mathbf{p}} = c_{\mathbf{p}} a_{\mathbf{p}} - s_{\mathbf{p}} a^*_{-\mathbf{p}}$$

$$s_{\mathbf{p}} := \mathrm{sign}(\widehat{V}(\mathbf{p})) \frac{\sqrt{\frac{|\mathbf{p}|^2}{2} + \nu \frac{\widehat{V}(\mathbf{p})}{\widehat{V}(\mathbf{0})} - \omega_{\mathrm{bg}}(\mathbf{p})}}{\sqrt{2\omega_{\mathrm{bg}}(\mathbf{p})}}, \qquad c_{\mathbf{p}} := \sqrt{1 + s^2_{\mathbf{p}}}$$

Ω_{bg} is the ground state in the $\mathscr{N}^{\mathrm{ext}} = \mathbb{1}N$ sector

$$\Omega_{\mathrm{bg}} = \prod_{\mathbf{p}\in\Xi^>_L} c_{\mathbf{p}}^{-\frac{1}{2}} \mathrm{e}^{-\frac{s^2_{\mathbf{p}}}{2c^2_{\mathbf{p}}} a^*_{\mathbf{p}} a^*_{-\mathbf{p}}} |N) \otimes \Omega^>$$

$\Omega^>$ is the vacuum in $\Gamma(\Xi^>_L)$ whereas $|N)$ is the $N-$ zero-mode particles state

Estimate of the tail terms

We define the Bogoliubov condensate density

$$\frac{1}{L^d}(\Omega_{\rm bg}|N_{\mathbf{0}}^{\rm ext}\Omega_{\rm bg}) = n_{0,\rm bg} = n - \frac{1}{L^d}\sum_{\mathbf{p}\in\Xi_L^>} s_{\mathbf{p}}^2$$

We assume $n_{0,\rm bg} > 0 \to$ this is always true if κ is small and $\nu \in [0,\nu_0]$, $\nu_0 > 0$
Then, one can prove

$$(b_{\mathbf{k}}^*\Omega_{\rm bg}|\,\mathbb{1}_{]-\infty,0]}(N_{\mathbf{0}}^{\rm ext})b_{\mathbf{k}}^*\Omega_{\rm bg}) \le e^{-cL^{d-1}}, \qquad c>0$$

Therefore **tail terms do not contribute** as $L\to+\infty$ to order κ

$$\lim_{L\to+\infty}(b_{\mathbf{k}}^*\Omega_{\rm bg}|R_{4,i}^L b_{\mathbf{k}}^*\Omega_{\rm bg}) = 0, \qquad i=1,2, \quad \textbf{(Hellmann-Feynman correction)}$$

Perturbation theory in the canonical formalism

The contributions to $\delta^{(2)}(\mathbf{k})$ come from two corrections: the Hellmann-Feynman term and the Fermi golden rule term

$$\delta^{(2)}(\mathbf{k}) := \lim_{\varepsilon \searrow 0} \lim_{L\to\infty} \Big\{ \big(b^*_{\mathbf{k}}\Omega_{\rm bg}|H^L_2 + H^L_{4,1} + H^L_{4,2} + H^L_{4,3} b^*_{\mathbf{k}}\Omega_{\rm bg}\big) \tag{1}$$

$$- \Big(H^L_3 b^*_{\mathbf{k}}\Omega_{\rm bg}|\big(H^L_{\rm bg} - \omega_{\rm bg}(\mathbf{k}) - \mathrm{i}\varepsilon\big)^{-1} H^L_3 b^*_{\mathbf{k}}\Omega_{\rm bg}\Big) \tag{2}$$

$$- \big(\Omega_{\rm bg}\,|H^L_4\Omega_{\rm bg}\big) + \big(H^L_3\Omega_{\rm bg}\,|(H^L_{\rm bg} - \mathrm{i}\varepsilon)^{-1} H^L_3\Omega_{\rm bg}\big)\Big\} \tag{3}$$

The counterterms remove a $O(L^d)$, $\mathbf{k}$-independent, term coming from the Bogoliubov vacuum $\Omega_{\rm bg}$

Perturbation theory in the canonical formalism

The counterterms can be absorbed into **commutators**

$$\begin{aligned}\delta^{(2)}(\mathbf{k}) = \kappa \lim_{\varepsilon \searrow 0} \lim_{L\to\infty} \Big\{ & (\Omega_{\rm bg}|[b_{\mathbf{k}}, [H_2^L + H_{4,1}^L + H_{4,2}^L + H_{4,3}^L, b_{\mathbf{k}}^*]]\Omega_{\rm bg}) \\ & - \big(\Omega_{\rm bg}|[b_{\mathbf{k}}, H_3^L](H_{\rm bg}^L - \omega_{\rm bg}(\mathbf{k}) - \mathrm{i}\varepsilon)^{-1}[H_3^L, b_{\mathbf{k}}^*]\Omega_{\rm bg}\big) \\ & - \big(\Omega_{\rm bg}|[b_{\mathbf{k}}^*, H_3^L](H_{\rm bg}^L + \omega_{\rm bg}(\mathbf{k}) - \mathrm{i}\varepsilon)^{-1}[H_3^L, b_{\mathbf{k}}]\Omega_{\rm bg}\big) \\ & - \big(\Omega_{\rm bg}|H_3^L(H_{\rm bg}^L - \mathrm{i}\varepsilon)^{-1}[b_{\mathbf{k}}, [H_3^L, b_{\mathbf{k}}^*]]\Omega_{\rm bg}\big) \\ & - \big(\Omega_{\rm bg}|[b_{\mathbf{k}}^*, [H_3^L, b_{\mathbf{k}}]](H_{\rm bg}^L - \mathrm{i}\varepsilon)^{-1}H_3^L\Omega_{\rm bg}\big) \Big\}\end{aligned}$$

Grand-canonical formalism: the Hamiltonian

We consider the second-quantized Hamiltonian

$$H^L_\mu := H^L - \mu \mathcal{N} = \sum_{\mathbf{p}\in\Xi_L} \left(\frac{|\mathbf{p}|^2}{2} - \mu \right) a^*_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{L^d} \sum_{\mathbf{p},\mathbf{q},\mathbf{k}\in\Xi_L} \widehat{V}(\mathbf{k}) a^*_{\mathbf{p}-\mathbf{k}} a^*_{\mathbf{q}+\mathbf{k}} a_{\mathbf{q}} a_{\mathbf{p}}$$

We **displace zero-modes** so to remove their "expectation value" in the ground state

$$W^*_{\mathbf{0}} a_{\mathbf{k}} W_{\mathbf{0}} = a_{\mathbf{k}} - \sqrt{\frac{\mu_0 L^d}{\kappa \widehat{V}(\mathbf{0})}} \delta_{\mathbf{k},0}, \qquad W_{\mathbf{0}} = \exp\left(- \sqrt{\frac{\mu_0 L^d}{\kappa \widehat{V}(\mathbf{0})}} a^*_{\mathbf{0}} + \sqrt{\frac{\mu_0 L^d}{\kappa \widehat{V}(\mathbf{0})}} a_{\mathbf{0}} \right)$$

The Hamiltonian becomes

$$W^*_{\mathbf{0}} H^L_\mu W_{\mathbf{0}} = H^L_{0,2} + H^L_{0,3} + \kappa H^L_{0,4} + E^L_0 + H^L_{\mathrm{bg},\mu_0} + \sqrt{\kappa} H^L_{3,\mu_0} + \kappa H^L_{4,\mu_0} - (\mu - \mu_0) N^>$$

Condensate density

- $H^L_{0,2} + H^L_{0,3} + H^L_{0,4} + E^L_0$ contain zero-modes $a_{\mathbf{0}}, a^*_{\mathbf{0}}$

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- $H^L_{3,\mu}$ and $H^L_{4,\mu}$ contain second-quantized interaction terms in $\Gamma(\Xi^>_L)$
- H^L_{bg,μ_0} is the free Bogoliubov Hamiltonian which can be diagonalized by $b^*_{\mathbf{k}}\Omega_{\mathrm{bg}}$ with dispersion relation

$$\Omega_{\mathrm{bg}} = \prod_{\mathbf{p}\in\Xi^>_L} c_{\mathbf{p}}^{-\frac{1}{2}} \mathrm{e}^{-\frac{s_{\mathbf{p}}^2}{2c_{\mathbf{p}}^2} a^*_{\mathbf{p}} a^*_{-\mathbf{p}}} W^*_{\mathbf{0}} |n_{0,\mathrm{bg}} L^d) \otimes \Omega^>, \qquad \omega_{\mathrm{bg},\mu}(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|^4}{4} + \mu_0 \frac{\widehat{V}(\mathbf{k})}{\widehat{V}(\mathbf{0})} |\mathbf{k}|^2}$$

and the condensate density $n_{0,\mathrm{bg}}$ is chosen as

$$n_{0,\mathrm{bg}} L^d := \frac{\mu_0 L^d}{\kappa \widehat{V}(\mathbf{0})}, \qquad (\Omega_{\mathrm{bg}} | W^*_{\mathbf{0}} a^*_{\mathbf{0}} a_{\mathbf{0}} W_{\mathbf{0}} \Omega_{\mathrm{bg}}) = n_{0,\mathrm{bg}} L^d$$

Fixing the condensate density

We choose $n_{0,\mathrm{bg}}$ (or equivalently μ_0) so to minimize the **perturbative ground state energy** $(\Omega_{\mathrm{bg}}|W_{\mathbf{0}}^* H_\mu^L W_{\mathbf{0}}\Omega_{\mathrm{bg}})$ Hugenholtz & Pines '59

$$\frac{\partial}{\partial\mu_0}(\Omega_{\mathrm{bg}}|W_{\mathbf{0}}^* H_\mu^L W_{\mathbf{0}}\Omega_{\mathrm{bg}}) = 0 \quad \Rightarrow \quad \mu_0 = \mu - \kappa\mu_1 + O(\kappa^2)$$

$$\mu_1 = \frac{\widehat{V}(\mathbf{0})}{L^d}\sum_{\mathbf{p}\in\Xi_L^>}\left[\left(1+\frac{\widehat{V}(\mathbf{p})}{\widehat{V}(\mathbf{0})}\right)s_{\mathbf{p}}^2 - \frac{\widehat{V}(\mathbf{p})}{\widehat{V}(\mathbf{0})}c_{\mathbf{p}}s_{\mathbf{p}}\right]$$

Fixing the condensate density

We choose $n_{0,\mathrm{bg}}$ (or equivalently μ_0) so to minimize the **perturbative ground state energy** $(\Omega_{\mathrm{bg}}|W_{\mathbf{0}}^* H_\mu^L W_{\mathbf{0}}\Omega_{\mathrm{bg}})$ Hugenholtz & Pines '59

$$\frac{\partial}{\partial\mu_0}(\Omega_{\mathrm{bg}}|W_{\mathbf{0}}^* H_\mu^L W_{\mathbf{0}}\Omega_{\mathrm{bg}}) = 0 \quad \Rightarrow \quad \mu_0 = \mu - \kappa\mu_1 + O(\kappa^2)$$

$$\mu_1 = \frac{\widehat{V}(\mathbf{0})}{L^d}\sum_{\mathbf{p}\in\Xi_L^>}\left[\left(1+\frac{\widehat{V}(\mathbf{p})}{\widehat{V}(\mathbf{0})}\right)s_{\mathbf{p}}^2 - \frac{\widehat{V}(\mathbf{p})}{\widehat{V}(\mathbf{0})}c_{\mathbf{p}}s_{\mathbf{p}}\right]$$

μ is chosen so to satisfy

$$n = \frac{1}{L^d}\left(\Omega_{\mathrm{bg}}|\mathscr{N}\Omega_{\mathrm{bg}}\right) = n_{0,\mathrm{bg}} + \frac{1}{L^d}\sum_{\mathbf{p}\in\Xi_L^>} s_{\mathbf{p}}^2$$

RMK: μ, μ_0 and ν **agree** at leading order but **differ** at order κ

Perturbation theory in the grand-canonical formalism

The Hellmann-Feynman and Fermi golden rule contributions are computed as

$$\begin{aligned}\widetilde{\delta}_\mu(\mathbf{k}) := \lim_{\varepsilon\searrow 0}\lim_{L\to\infty}\Big\{ & (\Omega_{\mathrm{bg}}|[b_{\mathbf{k}},[H^L_{4,\mu}-\mu_1 N^{>},b^*_{\mathbf{k}}]]\Omega_{\mathrm{bg}}) \\ & -\big(\Omega_{\mathrm{bg},\mu}|[b_{\mathbf{k}},H^L_{3,\mu}](H^L_{\mathrm{bg},\mu}-\omega_{\mathrm{bg},\mu}(\mathbf{k})-\mathrm{i}\varepsilon)^{-1}[H^L_{3,\mu},b^*_{\mathbf{k}}]\Omega_{\mathrm{bg}}\big) \\ & -\big(\Omega_{\mathrm{bg}}|[b^*_{\mathbf{k}},H^L_{3,\mu}](H^L_{\mathrm{bg},\mu}+\omega_{\mathrm{bg},\mu}(\mathbf{k})-\mathrm{i}\varepsilon)^{-1}[H^L_{3,\mu},b_{\mathbf{k}}]\Omega_{\mathrm{bg}}\big) \\ & -\big(\Omega_{\mathrm{bg}}|H^L_{3,\mu}(H^L_{\mathrm{bg},\mu}-\mathrm{i}\varepsilon)^{-1}[b_{\mathbf{k}},[H^L_{3,\mu},b^*_{\mathbf{k}}]]\Omega_{\mathrm{bg}}\big) \\ & -\big(\Omega_{\mathrm{bg}}|[b^*_{\mathbf{k}},[H^L_{3,\mu},b_{\mathbf{k}}]](H^L_{\mathrm{bg},\mu}-\mathrm{i}\varepsilon)^{-1}H^L_{3,\mu}\Omega_{\mathrm{bg}}\big)\Big\}\end{aligned}$$

Up to order $O(\kappa)$ we can substitute $\mu=\nu=\mu_0$ in the above

Canonical and grand-canonical corrections

There is an additional contribution to order $O(\kappa)$ coming from

$$\omega_{\mathrm{bg},\mu}(\mathbf{k}) = \omega_{\mathrm{bg},\mu}(\mathbf{k}) \restriction_{\mu_0=\nu} + (\mu_0 - \nu)\frac{\partial}{\partial \mu_0}\omega_{\mathrm{bg},\mu}(\mathbf{k}) \restriction_{\mu_0=\nu}$$

$$\delta_\mu^{(2)}(\mathbf{k}) = \widetilde{\delta}_\mu(\mathbf{k}) + (\mu_0 - \nu)\frac{\partial}{\partial \mu_0}\omega_{\mathrm{bg},\mu}(\mathbf{k}) \restriction_{\mu_0=\nu}$$

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- $H_{3,\mu}^L \restriction_{\mu_0=\nu} = H_3^L \rightarrow$ The fermi Golden rule coincides
- Hellmann-Feynman term differs

$$\begin{aligned}&\left(\Omega_{\mathrm{bg}}\right| [b_{\mathbf{k}}, [H_2^L + \ldots, b_{\mathbf{k}}^*]]\Omega_{\mathrm{bg}}\big) - (\Omega_{\mathrm{bg}}|[b_{\mathbf{k}}, [H_{4,\mu}^L - \mu_1 N^>, b_{\mathbf{k}}^*]]\Omega_{\mathrm{bg}}\big) \restriction_{\mu_0=\nu} \\ &= -\widehat{V}(\mathbf{k})(c_{\mathbf{k}} - s_{\mathbf{k}})^2 \frac{1}{L^d} \sum_{\mathbf{p}\in\Xi_L^>} s_{\mathbf{p}}^2 + O(L^{-d})\end{aligned}$$

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$$= -\widehat{V}(\mathbf{k})(c_{\mathbf{k}}-s_{\mathbf{k}})^2\frac{1}{L^d}\sum_{\mathbf{p}\in\Xi^{>}_L} s^2_{\mathbf{p}} + O(L^{-d}) = (\mu_0-\nu)\frac{\partial}{\partial\mu_0}\omega_{\mathrm{bg},\mu}(\mathbf{k}) \restriction_{\mu_0=\nu}$$

$\delta^{(2)}_{\mu}(\mathbf{k}) = \delta^{(2)}(\mathbf{k})$: canonical and grand-canonical coincide

Linear coefficient for $d = 3$

We can **explicitly** compute

$$\omega_{\mathrm{bg}}(\mathbf{k}) + \kappa\delta^{(2)}(\mathbf{k}) = |\mathbf{k}| C_{\mathrm{Blv}}(\nu, \widehat{V}) + o\left(\frac{|\mathbf{k}|}{\sqrt{\nu}}\right)$$

where

$$C_{\mathrm{Blv}}(\nu, \widehat{V}) = \sqrt{\nu} - \kappa\widehat{V}(0)\nu\left(\int_0^{+\infty} \frac{\mathrm{d}p}{32\pi^2\sqrt{\nu}} \frac{\widehat{V}(p)^2}{\widehat{V}(0)^2} \frac{p^6}{\omega_{\mathrm{bg}}(p)^3} + r_{\mathrm{Blv}}(\nu, \widehat{V})\right)$$

$$r_{\mathrm{Blv}}(\nu, \widehat{V}) = -\frac{1}{16\pi^2} + \int_0^{+\infty} \frac{\mathrm{d}p}{8\pi^2\nu}\Bigg[-\frac{ps_p^2}{12}\Big\{3(c_p - s_p)^6\big[(c_p - s_p)^2 + 2\big] - 2(c_p - s_p)^2\big[2(c_p - s_p)^2 + 7\big] + 17\Big\}$$

$$+ \sqrt{\nu}\frac{p\widehat{V}'(p)}{6\widehat{V}(0)} s_p(c_p + s_p)\Big\{(c_p - s_p)^4\big[1 + (c_p - s_p)^2\big]\big[3(c_p - s_p)^4 - 2\big] + 2\Big\}$$

$$- \nu\frac{p\widehat{V}''(p)}{12\widehat{V}(0)} s_p(c_p - s_p)\Big[(c_p - s_p)^4 + (c_p - s_p)^2 + 2\Big] + \nu^{\frac{3}{2}}\frac{\widehat{V}'(p)^2}{3\widehat{V}(0)^2} c_p s_p(c_p - s_p)^4\Big[3(c_p - s_p)^4 + 5\Big]\Bigg]$$

Dilute limit

The correction $\delta^{(2)}(\mathbf{k})$ is linear in $\mathbf{k}$ for small momenta as predicted in Beliaev '57, Hugenholtz & Pines '59 but the **coefficient** $C_{\mathrm{Blv}}(\nu, \widehat{V})$ appears to be different

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- Replace $C_{\mathrm{Blv}}(\nu, \widehat{V}) \to C_{\mathrm{Blv}}(\kappa\nu, \widehat{V})$ the first two terms can be re-organized as

$$\begin{aligned}
&\sqrt{\kappa\nu} - \kappa^{\frac{3}{2}}\widehat{V}(0)\sqrt{\nu}\int_0^{+\infty}\frac{\mathrm{d}p}{32\pi^2}\frac{\widehat{V}(p)^2}{\widehat{V}(0)^2}\frac{p^6}{\omega_{\mathrm{bg}}(p)^3}\\
=&\sqrt{\kappa\nu} - \kappa\widehat{V}(0)\sqrt{\nu}\int_0^{+\infty}\frac{\mathrm{d}p}{4\pi^2}\frac{\widehat{V}(\sqrt{\nu\kappa}p)^2}{\widehat{V}(0)^2}\left(\frac{p^6}{(p^4+4p^2\widehat{V}(\sqrt{\nu\kappa}p))^{\frac{3}{2}}} - 1\right)\\
&- \kappa^{\frac{3}{2}}\widehat{V}(0)\sqrt{\nu}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{\widehat{V}(\mathbf{p})^2}{2|\mathbf{p}|^2\widehat{V}(0)^2} = \sqrt{4\pi n\mathfrak{a}}\left(1 + \frac{1}{\pi^2}\sqrt{n(4\pi\mathfrak{a})^3}\right)(1 + O(\sqrt{\kappa})).
\end{aligned}$$

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\end{aligned}$$

- The **remainder** $r_{\text{Blv}}(\kappa\nu, \widehat{V}) = O(\kappa^2)$ due to **derivatives** of the potential $\widehat{V}', \widehat{V}''$ and **cancellations**. Similarly, we recover Beliaev's result for $C_{\text{Blv}}(\nu, \widehat{V}) = C_{\text{Blv}}(\nu, \widehat{V}(0))$

- second-order perturbation theory is consistent between the **grand-canonical** and **canonical** formalism
- The shift is linear in $\mathbf{k}$ for small $\frac{\mathbf{k}}{\sqrt{\nu}} \to \mathbf{0}$ coherently with Hugenholtz & Pines '59
- We recover Beliaev '57 result in the **dilute limit** $\nu \to 0$

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Thank you for your attention!